CONVERGENCE THEOREMS FOR VECTOR INTEGRALS

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In this dissertation we develop several theorems concerning asymptotic martingales, or amarts, in two aspects. First we treat real-valued asymptotic martingales which are integrable with respect to a vector-valued measure whose values are in a locally convex topological vector space. Secondly, we treat Banach-space (or B-space)-valued asymptotic martingales which are integrable with respect to a B-space-valued measure. In Chapter I, we establish basic concepts concerning real-valued asymptotic martingales integrable with respect to a real-valued measure. In Chapter II, the uniform integrability and the decomposition of real-valued amarts integrable with respect to a vector-valued measure whose values are in a locally convex topological vector space are derived. Also convergence theorems of real-valued net amarts integrable with respect to a vector-valued measure which takes values in a locally convex topological vector space are derived. In Chapter III, several theorems concerning B-space-valued martingales integrable with respect to a B-space-valued measure are developed. We also extend some of these theorems and an optional sampling theorem to the amart case.

CHAPTER I

REAL-VALUED ASYMPTOTIC MARTINGALES

1. Introduction

A sequence of integrable functions (f_n) defined on a probability space (S, Σ, p) and adapted to an increasing sequence of sub-σ-fields (Σ_n) is called an asymptotic martingale, abbreviated as amart, if $\int |f_n| dp < \infty$ for all $n \in N$ and $\lim_{\tau \in T} \int f_\tau dp$ exists, where N is the set of all natural numbers, and T is the collection of all bounded stopping times for (Σ_n), the limit on T is with respect to the usual ordering on T. A martingale is a special case of amart.

The concept of an amart was first given by Meyer [9] who proved that a continuous parametered scalar-valued amart converges almost everywhere if it is essentially bounded. Austin, Edgar, and Tulcea [1] proved that (*) "a real-valued amart converges almost everywhere if it is L_1-bounded." Chatterji [5] proved that if a Banach space E has the Radon-Nikodým property, then every E-valued L_1-bounded martingale converges almost everywhere. Chacon and Sucheston [4] proved that if E is a Banach space with a separable dual and has the Radon-Nikodým property, then an E-valued amart (f_n) converges almost everywhere in the weak topology of E if the condition $\sup_{\tau \in T} \int |f_\tau| dp < \infty$ holds. They also showed that strong convergence need not hold. Smith [11] developed some convergence properties for weakly measurable Pettis integrable amarts as well as for strongly measurable Pettis integrable amarts which take values in a locally convex topological vector space V. In this dissertation we continue to develop the convergence properties in a general vector space.

This dissertation is concerned with two topics. The first topic is the convergence properties for scalar-valued amarts in which functions are integrable with respect to a measure which takes values in a locally convex topological vector space. In this topic amarts are integrable in the sense of the integral defined by Lewis [8].

In the second topic, we first develop properties which pertain to martingales whose values are in a Banach space X and integrable with respect to a measure μ which takes values in a (possibly different) Banach space Y such that there is a continuous bilinear "multiplication" defined on the product of these two spaces X, Y and the product lying in a (possibly different) Banach space Z. After this, we develop these extended properties to the amart case.

In the first topic, convergence theorems were partially treated in the dissertation of Robert W. Smith. We will continue this treatment in three aspects. First we show that an amart (f_n) converges to an integrable function in L_1 if and only if (f_n) is uniformly integrable, provided V is sequentially complete. The proof of the real-valued martingale case can be found in [10, p. 65]. Next we show the "Riesz decomposition" of an amart into the sum of a martingale and an amart. The proof of the real-valued amart case can be found in [7, p. 209]. Finally, we show some convergence theory of a net amart which converges in measure under certain conditions. The proof is developed along the line of the corresponding theory of scalar-valued amarts developed in [7, p. 206].

In the second topic, amarts are integrable in the sense of the integral defined by Bartle [2]. We develop some properties concerning martingales. Some of the proofs are developed along the lines of the corresponding theorems of real-valued amarts developed in [1, p. 18]. After this, we extend these properties to the amart case. However, we cannot extend the convergence theorem (*) in this topic, because it is not true in general that if two μ-integrable functions f and g satisfy $\int_A f d\mu = \int_A g d\mu$ for every set A in the σ-field Σ, then $f = g$ μ-almost everywhere.

2. Elementary Notations

We shall begin by introducing basic concepts and notations for the real-valued amarts which are integrable with respect to a non-negative measure λ. A field of subsets of S, or a Boolean algebra of subsets of S, is a non-empty family of subsets of S which contains the empty set, the complement (relative to S) of every element, and the union of any finite collection of its elements. A σ-field or Borel-field of S is a field which contains the union of any countable collection of its elements. The pair (S, Σ) consisting of a set S and a σ-field Σ of subsets of S is called a measurable space. A function $\lambda: \Sigma \rightarrow [0, \infty)$ is a countably additive set function on Σ if for every disjoint sequence (E_n) is Σ with $\bigcup_{n=1}^{\infty} E_n = E$, $\lambda(E) = \sum_{n=1}^{\infty} \lambda(E_n)$ where the convergence of the infinite series is unconditional.

A function $f: S \rightarrow R$ is Σ-measurable if for every real value a, the set $\{s \varepsilon S : f(s) < a\}$ is in Σ. For convenience, we will denote this set by $(f < a)$. If f is a Σ-measurable function defined on S, then the σ-field generated by f is the smallest σ-field of S which contains all sets of the form $(f < a)$ for any real value a. Similary, the sub-σ-field of Σ generated by a family of Σ-measurable functions $(f_\alpha, \alpha \varepsilon \Lambda)$, denoted by $\sigma\{(f_\alpha, \alpha \varepsilon \Lambda)\}$, is the smallest σ-field of S which contains all sets of the form $(f_\alpha \varepsilon B)$ where B is a Borel subset of R and α is in Λ.

A subset E of S is a λ-null set if E is contained in some set F of Σ such that $\lambda(F) = 0$. Any statement concerning the points of S is said to hold λ-almost everywhere, or simply almost everywhere, if it is true except for those elements in some λ-null set. The phrase "almost everywhere" is usually abbreviated "a.e." A function $f: S \rightarrow R$ is said to be λ-essentially bounded if

$$\inf_{N} \sup_{s \in S-N} |f(s)| < \infty$$

where the infimum is taken over all λ-null sets N. A function $f: S \to R$ is a λ-null function if $(|f| > a)$ is a λ-null set, $a > 0$. If the function $f: S \to R$ takes only a finite number of values $x_1, x_2, \ldots, x_n$ and for which the sets

$f^{-1}(x_i) = \{s: s \in S, f(s) = x_i\}$ lie in Σ, $i = 1,2,3,\ldots,n$,

then any function g from S to R which differs from f by a λ-null function is called a λ-simple function. A sequence (f_n) of functions from S to R is said to converge in λ-measure, or converge in measure, to a function f from S to R if

$$\operatorname{limit}_{n} \lambda(s: |f_n(s) - f(s)| > \varepsilon) = 0 \quad \text{for every } \varepsilon > 0.$$

A function f on S to R is a λ-measurable function on S if there is a sequence of λ-simple functions converging to f in measure.

Two other convergence concepts for sequences which are often used in real-valued measurable functions are almost everywhere convergence and convergence in mean. We say that a sequence (f_n) converges to f almost everywhere, $f_n \to f$ a.e., if there is a λ-null set N such that $\operatorname{limit}_{n\to\infty} f_n(s) = f(s)$ for every s in S - N. Let (f_n) be a sequence of L_1-integrable functions, we say that (f_n) converges to f in mean if the sequence (f_n) converges to f in L_1-norm. This means $\operatorname{limit}_{n} \int |f_n - f| d\lambda = 0$. If $\lambda(S) = 1$, then (S, Σ, λ) is called a probability space. On a probability space, a Σ-measurable function will be called a random variable, abbreviated as r.v. A random variable is also a measurable function on that probability space.

3. Conditional Expectation of a Random Variable

From now on we will use (S, Σ, p) to denote a probability space. For any sub-σ-field J of Σ, the countably additive set function $\nu: J \longrightarrow R$ is said to be absolutely continuous with respect to P, denoted by $\nu << P$ if $\nu(E) \longrightarrow 0$ whenever $P(E) \longrightarrow 0$. Now we state the well-known Radon-Nikodým theorem:

Let (S, Σ, λ) be a finite positive measure space, and ν a finite positive measure on Σ, which is absolutely continuous with respect to λ. Then there exists a unique function f in $L_1(S, \Sigma, \lambda)$ such that $\nu(E) = \int_E f d\lambda$ for each E in Σ.

Let f be a random variable defined on S such that $\int |f| dp < \infty$, and J be a sub-σ-field of Σ. Then the set function $\nu : J \longrightarrow R$ defined by $\nu(E) = \int_E f dp$ for $E \varepsilon J$ is a countably additive set function and is absolutely continuous with respect to p. By the Radon-Nikodým theorem, there is a J-measurable, integrable function g such that $\int_E f dp = \int_E g dp$ for every E in J. A J-measurable function which differs from g by a null function is called the conditional expectation of f given J, and is denoted by $E(f|J)$. The conditional expectation of f given by $(f_\alpha: \alpha \varepsilon \Lambda)$ means the conditional expectation of f given the sub-σ-field of Σ generated by $(f_\alpha: \alpha \varepsilon \Lambda)$, denoted by $E(f/f_\alpha : \alpha \varepsilon \Lambda)$. Intuitively

$E(f/f_\alpha : \alpha \in \Lambda)$ is the "best estimate" of the random variable f given information from the random variables $\{f_\alpha : \alpha \in \Lambda\}$.

4. Martingales

Let (f_n) be a sequence of random variables and (Σ_n) an increasing sequence of sub-σ-fields of Σ. We say that (f_n) is adapted to (Σ_n) if f_n is a Σ_n-measurable function for each n in N. A sequence of random variables (f_n) which is adapted to (Σ_n) is called a martingale if $\int|f_n|dp < \infty$ for each $n \in N$ and $E(f_n/\Sigma_m) = f_m$ for each $m,n \in N$, $n \geq m$ or equivalently, $\int_E f_n dp = \int_E f_m dp$ for each E in Σ_m, $n,m \in N$ and $n \geq m$. A martingale is said to be L_1-bounded if $\sup_n \int|f_n|dp < \infty$.

For any martingale $(f_n, \Sigma_n, n \in N)$ the following conditions are equivalent:

(a) The sequence (f_n) converges in L_1.

(b) There exists an integrable random variable f such that $f_n = E(f/\Sigma_n)$ for all n in N.

(c) The sequence $(f_n, n \in N)$ satisfies the uniform integrability condition, that is,

$$\lim_{a\to\infty} \int_{(|f_n|>a)} |f_n|dp = 0 \text{ uniformly in } n \in N.$$

The proof of the preceding property can be found in [10, p. 65]. A martingale is called a regular martingale if it satisfies one of these equivalent conditions.

Let $(f_n, \Sigma_n, n \in N)$ be a sequence of random variables adapted to an increasing sequence (Σ_n) of sub-σ-fields of Σ. We call (f_n) a submartingale if $\int |f_n| dp < \infty$ for each n and $E(f_n/\Sigma_m) \geq f_m$, or $\int_E f_n dp \geq \int_E f_m dp$, for $n, m \in N$, $n \geq m$, $E \in \Sigma_m$. (f_n) is a supermartingale if $\int |f_n| dp < \infty$ for each $n \in N$ and $E(f_n/\Sigma_m) \leq f_m$, or $\int_E f_n dp \leq \int_E f_m dp$, for $n, m \in N$, $n \geq m$, $E \in \Sigma_m$. Note that $(f_n, \Sigma_n, n \in N)$ is a supermartingale if and only if $(-f_n, \Sigma_n, n \in N)$ is a submartingale. We say that a sequence of random variables $(Z_n, n \in N)$ is an increasing process if it satisfies the conditions

(1) $Z_1 = 0$; $Z_n \leq Z_{n+1}$ for $n \geq 1$;

(2) $E(Z_n) < \infty$ for each n, where $E(Z_n) = \int Z_n dp$.

We state the well-known Doob's decomposition of positive supermartingales:

Every finite positive supermartingale $(f_n, n \in N)$ can be written in one and only one way as the difference between a finite positive martingale $(g_n, n \in N)$ and an increasing process $(h_n, n \in N)$, and $f_n = g_n - h_n$ for each $n \in N$. The proof of this property can be found in [10, p. 171].

5. Asymptotic Martingales

Throughout this section we will continue to assume that (f_n) is a sequence of random variables adapted to an increasing sequence (Σ_n) of sub-σ-fields. A random variable $\tau: S \rightarrow N$ is a stopping time for (Σ_n) if $(\tau=n) \in \Sigma_n$ for each n in N.

Let T be the collection of all bounded stopping times for (Σ_n). For each $\tau \varepsilon T$, we define a random variable f_τ by $f_\tau(s) = f_{\tau(s)}(s)$, that is, $f_\tau(s) = f_n(s)$ for each s in the set $(\tau=n)$, $n \varepsilon N$. Let τ_1, τ_2 be two bounded stopping times. We say that τ_1 is not less than τ_2, denoted by $\tau_1 \geq \tau_2$, if $\tau_1(s) \geq \tau_2(s)$ almost everywhere. A partially ordered set $(\Lambda, \leq)$ is said to be directed if every finite subset of Λ has an upper bound.

$(f_n, \Sigma_n, n \varepsilon N)$ is an asymptotic martingale, or simply amart, if $\int|f_n|dp < \infty$ for all $n \varepsilon N$ and $\lim_{\tau \varepsilon T} \int f_\tau dp$ exists. A martingale $(f_n, \Sigma_n, n \varepsilon N)$ is also an amart. This fact is proved in the following manner:

Let τ be any bounded stopping time for (Σ_n). Then there is an integer r such that $\tau \leq r$. Therefore

$$\int f_\tau dp = \sum_{n=1}^{r} \int_{(\tau=n)} f_n dp = \sum_{n=1}^{r} \int_{(\tau=n)} f_r dp = \int f_r dp$$

$$= \int f_1 dp.$$

Hence $\int f_\tau dp$ does not depend on the choice of $\tau \varepsilon T$. Thus f(n) is an amart. Similarly submartingales and supermartingales are amarts if $(\int f_n dp, n \varepsilon N)$ is bounded.

As in Doob's decomposition of positive supermartingales, there is a decomposition for amarts given as follows:

Let $(f_n, \Sigma_n, n \varepsilon N)$ be an amart. Then f_n can be uniquely written as $f_n = g_n + h_n$, where $(g_n, \Sigma_n, n \varepsilon N)$, or simply denoted by (g_n) if $(\Sigma_n, n \varepsilon N)$ is fixed, is a martingale.

and (h_n) is an amart with $h_n \to 0$ in L_1. In addition, $(h_\tau, \tau \in T)$ is uniformly integrable and $h_n \to 0$ a.e. For the proof of this decomposition theorem, see [7, p. 209].

CHAPTER II

REAL-VALUED ASYMPTOTIC MARTINGALES INTEGRABLE WITH RESPECT TO A VECTOR-VALUED MEASURE

1. Basic Background and Uniform Integrability

In this chapter we shall continue to develop the theory of real-valued amarts which was treated partially in [11]. These amarts are integrable in the sense of the integral defined by Lewis [8]. Lewis developed the integration theory chiefly through the study of the P-semi-variation of the vector measure whose values are in a locally convex topological vector space V, where P is a semi-norm on V.

We shall start by introducing the theory of integration developed by Lewis, and the properties of real-valued amarts which are integrable with respect to a V-valued measure μ, investigated by Smith in [11].

Throughout this chapter (S, Σ) will denote a measure space, V a locally convex topological vector space, and μ a V-valued countably additive set function on Σ. A continuous function P from V into R satisfying

(1) $0 \leq P(x) < +\infty$,

(2) $P(\alpha x) = |\alpha| P(x)$,

and (3) $P(x+y) \leq P(x) + P(y)$

for any x, y ε V and any complex number α, is called a continuous semi-norm P defined on V. If $x^* \in V^*$ and P is a continuous semi-norm defined on V we will write $x^* \leq P$ whenever $|x^*(x)| \leq P(x)$ for all $x \in X$. If P is a semi-norm on V, then the P-semivariation of μ is the function from Σ into the extended reals defined by

$$\|\mu\|_p(E) = \sup \{v(x^*\mu, E) : x^* \leq P\}$$

where $v(x^*\mu, \cdot)$ is the scalar variation of $x^*\mu$, that is,

$$v(x^*\mu, E) = \sup \sum_{i=1}^{n} |x^*\mu(E_i)|$$

where the supremum is taken over all finite sequence $\{E_i\}$ of disjoint subsets of E in Σ.

For proofs of the next three results, see Lewis [8].

Proposition 2.1-1 [8, p. 158]. If μ is a countably additive measure and P is a continuous semi-norm on V, then $\|\mu\|_p(\cdot)$ is monotone, countably subadditive, real valued, and

$$P[\mu(E)] \leq \|\mu\|_p(E) \leq 4 \sup \{P[\mu(F)] : F \leq E, F \in \Sigma\}.$$

Theorem 2.1-2 [8, p. 158]. If μ is a measure, P a continuous semi-norm and $(E_n, n \in N)$ a convergent sequence of sets in Σ, then

$$\|\mu\|_p (\operatorname*{limit}_n E_n) = \operatorname*{limit}_n \|\mu\|_p (E_n).$$

Corollary 2.1-3 [8, p. 158]. If μ is a measure and $(E_n, n \in N)$ is a convergent sequence in Σ, then

$$\mu(\operatorname*{limit}_n E_n) = \operatorname*{limit}_n \mu(E_n).$$

A subset of S is called a $\underline{\mu\text{-null}}$ set if it is a subset of a set $E \varepsilon \Sigma$ such that $||\mu||_p(E) = 0$ for every continuous semi-norm P. Any statement concerning the elements of S is said to hold μ-almost everywhere, or simply almost everywhere if it is true except for those elements in a μ-null set.

A μ-simple function $f: S \to R$ is a function which is μ-almost everywhere equal to a function

$$g: S \to R,\ g(x) = \sum_{i=1}^{n} c_i I_{E_i}(x)$$

where $c_i \varepsilon R$ and $E_i \varepsilon \Sigma$ for each $i = i,2,\ldots,n$, $\bigcup_{i=1}^{n} E_i = S$ and $I_{E_i}(\cdot)$ is the characteristic function of E. A function $f : S \to R$ is μ-measurable if there exists a sequence of μ-simple functions which converge to f μ-almost everywhere. A function $f : S \to R$ is μ-integrable if

(1) f is $x^*\mu$-integrable for each $x^* \varepsilon X^*$ and

(2) for each $E \varepsilon \Sigma$ there is an element of X, denoted by $\int_E f d\mu$, such that $x^*(\int_E f d\mu) = \int_E f d(x^*\mu)$ for each x^* in V^*. Since the topology on V is Hausdorff the integral is well-defined. It is obvious that the integral is linear and for every simple function $f(x) = \sum_{i=1}^{n} c_i I_{E_i}$, f is μ-integrable and $\int_E (\sum_{i=1}^{n} c_i I_{E_i}) d\mu = \sum_{i=1}^{n} c_i \mu(E \cap E_i)$ for $E \varepsilon \Sigma$. If f is bounded and μ-integrable, then

$$P[\int_E f d\mu] \leq ||\mu||_p(E) \cdot \sup \{f(s) : s \varepsilon S\}$$

for each $E \varepsilon \Sigma$ and continuous semi-norm P.

Theorem 2.1-4 [8, p. 160]. (1) If f is μ-integrable, then the set function on Σ defined by $\Phi(E) = \int_E f d\mu$ is a measure,

$$||\Phi||_p(E) = \sup \{\int_E |f(t)| v(x^*\mu, dt) : x^* \leq P\}$$

and $\lim_{||\mu||_p(E) \to 0} ||\Phi||_p(E) = 0$ for each continuous semi-norm P.

(2) Let (f_n) be a sequence of μ-integrable functions which converge pointwise to f on S and g be a μ-integrable function such that $|f_n| \leq g$ for each n. Then f is μ-integrable if V is sequentially complete. If f is μ-integrable, then $\int_E f d\mu = \lim_n \int_E f_n d\mu$ uniformly with respect to $E \in \Sigma$.

In Theorem 2.1-4 we can replace the hypothesis "(f_n) converges pointwise to f on S" with "(f_n) converges to f μ-a.e. on S." The result is still true, because for each $x^* \in V^*$ there is a continuous semi-norm P on V such that $x^* \leq P$. This P can be chosen as $P: V \longrightarrow R^+$, $P(x) = |x^*(x)|$ for each $x \in V$. Therefore f is $x^*\mu$-integrable for each $x^* \in V^*$. The rest of the proof is the same as the proof in [8, p. 160].

Definition 2.1-5 [11, p. 60]. If $(f_n, \Sigma_n, n \in N)$ is a sequence of μ-measurable functions adapted to an increasing sequence of sub-σ-fields (Σ_n) of Σ, then $(f_n, \Sigma_n, n \in N)$ is defined to be an amart if

$$\sup \{\int |f_n| dv(x^*\mu) : x^* \leq P\} < \infty$$

for each $n \in N$ and each continuous semi-norm P, and $\lim_{\tau \in T} f_\tau d\mu$ exists in V. The existence of this limit means that there is a vector $x \in V$ such that for each $\varepsilon > 0$ and each continuous semi-norm P there is a $\sigma \in T$ such that if $\tau \geq \sigma$ then $\sup \{|x^*(\int f_\tau d\mu - x)| : x^* \leq P\} < \varepsilon$.

Lemma 2.1-6 [11, p. 61]. Let (f_n) be a sequence of μ-integrable functions adapted to an increasing sequence of sub-σ-fields (Σ_n) of Σ, and $\Phi_\tau(E) = \int_E f_\tau d\mu$ for each $\tau \in T$. If $\sup \{||\Phi_\tau(S)||_p : \tau \in T\} < \infty$ for every continuous semi-norm P, then for $\lambda > 0$

$$||\mu||_p(\{s\ S: \sup_n |f_n(s)| > \lambda\}) \leq \frac{1}{\lambda}(\sup \{\int |f_\tau| dv(x^*\mu):$$

$\tau \in T, x^* \leq P$) for every continuous semi-norm P.

Proposition 2.1-7 [11, p. 62]. If $(f_n, \Sigma_n, n \in N)$ is an amart then $\sup \{|\int f_\tau d(x^*\mu)| : x^* \leq P, \tau \in T\} < \infty$ for every continuous semi-norm P.

Proposition 2.1-8 [11, p. 65]. For each $E \in \Sigma$, the restriction of the sequence (f_n) to E is an amart for $(\Sigma_n \cap E : n \in N)$, where $\Sigma_n \cap E$ denotes the σ-field consisting of all intersections of elments of Σ_n with E.

Definition 2.1-9 [11, p. 69]. A sequence (f_n) of μ-integrable functions is said to be L_1-bounded if

$$\sup \{\int |f_n| dv(x^*\mu) : x^* \leq P;\ n \in N\} < \infty$$

for every continuous semi-norm P.

Theorem 2.1-10 [11, p. 64]. If V is a sequentially complete space and if $(f_n, \Sigma_n, n \in N)$, $(g_n, \Sigma_n, n \in N)$ are amarts, then $(f_n \vee g_n, \Sigma_n, n \in N)$ and $(f_n \wedge g_n, \Sigma_n, n \in N)$ are also amarts, where $(f_n \vee g_n)(s) = \max \{f_n(s), g_n(s)\}$ and $(f_n \wedge g_n)(s) = \min \{f_n(s), g_n(s)\}$ for each s in S.

Proposition 2.1-11 [11, p. 70]. If V is a sequentially complete space and if $(f_n, \Sigma_n, n \in N)$ is an L_1-bounded amart, then:

(a) $(|f_n|, \Sigma_n, n \in N)$, $(f_n^+, \Sigma_n, n \in N)$, $(f_n^-, \Sigma_n, n \in N)$ and $(-\alpha \vee f_n \wedge \alpha, \Sigma_n, n \in N)$ are also L_1-bounded amarts, where $\alpha \geq 0$.

(b) $\text{Sup}\ \{|\int |f_\tau| d(x^*\mu)|: \tau \in T, x^* \leq P\} < \infty$.

(c) $\text{Sup}\ \{|f_n|: n \in N\} < \infty$ μ-almost everywhere.

Proposition 2.1-12 [11, p. 72]. Let (f_n) be a sequence of μ-measurable functions adapted to $(\Sigma_n, n \in N)$ and suppose there is an integrable μ-measurable function g such that $|f_n| \leq g$ for all $n \in N$. Then if V is sequentially complete, the following are equivalent:

(a) $(f_n, \Sigma_n, n \in N)$ is an amart;

(b) the sequence (f_n) converges almost everywhere to a μ-integrable function f.

Lemma 2.1-13 [11, p.73]. Let $(f_n, \Sigma_n, n \in N)$ be an amart and assume that V is sequentially complete. Then the sequence (f_n) converges to a μ-integrable function f μ-a.e. if

$$\sup\ \{\int |f_n| dv(x^*\mu): n \in N, x^* \leq P\} < \infty$$

for every continuous semi-norm P.

The next theorem constitutes an "optional sampling theorem" for amarts.

Theorem 2.1-14 [11, p. 74]. Let $(f_n, \Sigma_n, n \in N)$ be an amart and $(\tau_n, n \in N)$ be a non-decreasing sequence of bounded stopping times for $(\Sigma_n, n \in N)$. Define $g_n = f_{\tau_n}$ and

$$\Sigma'_k = \Sigma_{\tau_k} = \{B \in \Sigma \colon B \cap \{\tau_k = n\} \in \Sigma_n, n \in N\}.$$

Then $(g_n, \Sigma'_n, n \in N)$ is an amart if V is sequentially complete.

Lemma 2.1-15 [11, p. 76]. If $(f_n, \Sigma_n, n \in N)$ is an amart and σ is a stopping time, then $(f_{n\wedge\sigma}, \Sigma_n, n \in N)$ is an amart if V is sequentially complete.

Proposition 2.1-16 [11, p. 77]. Let V be a sequentially complete space. If $(f_n, \Sigma_n, n \in N)$ is an amart and $(\tau_k, k \in N)$ is an increasing sequence of bounded stopping times with $\tau_k \geq k$, and if for each continuous semi-norm P

$$\sup \{\int (\sup_{k \in N} |f_{\tau_k} - f_{k-1}|) \, dv(x^*\mu) \colon x^* \leq P\} < \infty,$$

then (f_n) converges a.e. on the set $B = \{\sup_n |f_n| < \infty\}$.

Now we continue our development of the theory of amarts which has been illustrated in preceding paragraphs.

Definition 2.1-17. Let (f_n) be a sequence of μ-integrable functions and $\Phi_n(E) = \int_E f_n d\mu$ for each $n \in N$, $E \in \Sigma$. Then (f_n) is defined to be uniformly absolutely continuous with respect to μ if

$$\lim_{||\mu||_p(E) \to 0} ||\Phi_n||_p(E) = 0$$

uniformly in n, for every continuous semi-norm P, where

$$||\Phi_n||_p(E) = \sup \{\int_E |f_n| dv(x^*\mu) \colon x^* \leq P\}.$$

Definition 2.1-18. Let (f_n) be a sequence of μ-integrable functions. Then (f_n) is defined to be uniformly integrable if $\lim\limits_{N\to\infty} \|\Phi_n\|_P(|f_n| > N) = 0$ uniformly in n, for every continuous semi-norm P.

The next theorem shows that uniform integrability is equivalent to L_1-boundedness and uniform absolute continuity for a sequence of μ-integrable function.

Theorem 2.1-19. Let (f_n) be a sequence of μ-integrable functions. Then the following are equivalent:

(a) (f_n) is L_1-bounded and uniformly absolutely continuous with respect to μ;

(b) (f_n) is uniformly integrable.

Proof. First we shall show that (a) implies (b). Let ε be an arbitrary positive number and P an arbitrary but fixed continuous semi-norm on V. Since (f_n) is uniformly absolutely continuous with respect to μ, there is a number $\delta > 0$ such that $\|\Phi_n\|_P(E) < \varepsilon$ whenever $E \in \Sigma$ and $\|\mu\|_P(E) < \delta$. Since (f_n) is L_1-bounded, define

$$M = \sup \{\int |f_n| dv(x^*\mu) : x^* \leq P,\ n \in N\},$$

then M is finite.

Choose an integer N_o such that $N_o > M/\delta$. We claim that $\|\mu\|_P(|f_n| > N_o) < \delta$ holds for each $n \in N$. If this is not true, then there is some integer n_o such that $\|\mu\|_P(|f_{n_o}| > N_o) \geq \delta$.

Since

$$||\mu||_p(|f_{n_o}| > N_o) = \sup\{v(x^*\mu, (|f_{n_o}| > N_o)) : x^* \leq P\},$$

there is some $x_o^* \varepsilon V^*$, $x^* \leq P$ such that $v(x_o^*\mu, (|f_{n_o}| > N_o)) \geq \delta$.

Then

$$\int_{(|f_{n_o}|>N_o)} |f_{n_o}|dv(x_o^*\mu) \geq N_o \cdot v(x_o^*\mu, (|f_{n_o}|>N_o)) > M.$$

This is impossible because

$$M = \sup\{\int|f_n|dv(x^*\mu) : x^* \leq P,\ n \ \varepsilon\ N\}.$$

Therefore $||\mu||_p(|f_n| > N_o) < \delta$ is true for each $n\ \varepsilon\ N$.

This implies $||\Phi_n||_p(|f_n| > N_o) < \varepsilon$ for each $n\ \varepsilon\ N$.

Thus we have proved that (a) implies (b). Next we want to show that (b) implies (a). We still let P be an arbitrary but fixed continuous semi-norm.

Since (f_n) is uniformly integrable,

$$\lim_{N\to\infty} ||\Phi||_p(|f_n| > N) = 0 \text{ uniformly in } n.$$

Given $\varepsilon_o = 1$ there is an integer N_1 such that $||\Phi_n||_p((|f_n| > N_1)) < 1$ for each $n\ \varepsilon\ N$ where

$$||\Phi||_p((|f_n| > N_1) = \sup\{\int_{(|f_n|>N_1)} |f_n|dv(x^*\mu) : x^* \leq P\}.$$

Then

$$\int|f_n|dv(x^*\mu) = \int_{(|f_n|>N_1)}|f_n|dv(x^*\mu) + \int_{(|f_n|\leq N_1)}|f_n|dv(x^*\mu)$$

$$< 1 + N_1 \cdot v(x^*\mu, (|f_n| \leq N_1)) \leq 1 + N_1 \cdot v(x^*\mu, S)$$

$$\leq 1 + N_1 \cdot ||\mu||_p(S), \text{ for each } n\ \varepsilon\ N \text{ and each } x^* \leq P.$$

It follows from Proposition 2.1-1 that $||\mu||_p(S)$ is finite.

Therefore

$$\sup \{\int |f_n| dv(x^*\mu) : n \in N,\ x^* \leq P\} < 1 + N_1 \|\mu\|_p(S) < \infty.$$

This means that (f_n) is L_1-bounded. Let $\varepsilon > 0$ be given. Since (f_n) is uniformly integrable, there exists an integer N_ε such that

$$\|\Phi_n\|_p(|f_n| > N_\varepsilon) < \frac{\varepsilon}{2} \text{ for each } n \in N.$$

Choose $\delta = \frac{\varepsilon}{2N_\varepsilon}$ and let E be in Σ such that $\|\mu\|_p(E) < \delta$.

Then

$$\int_E |f_n| dv(x^*\mu) = \int_{E\cap(|f_n|>N_\varepsilon)} |f_n| dv(x^*\mu) + \int_{E\cap(|f_n|\leq N_\varepsilon)} |f_n| dv(x^*\mu)$$

$$\leq \int_{(|f_n|>N_\varepsilon)} |f_n| dv(x^*\mu) + N_\varepsilon \cdot v(x^*\mu, E) < \frac{\varepsilon}{2} + N_\varepsilon \cdot \|\mu\|_p(E) < \varepsilon$$

for each $n \in N$ and for each $x^* \in V^*$, $x^* \leq P$.

Therefore

$$\|\Phi_n\|_p(E) = \sup \{\int_E |f_n| dv(x^*\mu) : x^* \leq P\} < \varepsilon$$

for each $n \in N$. This means that given any $\varepsilon > 0$, there is some $\delta > 0$ such that $\|\Phi_n\|_p(E) < \varepsilon$ for each $n \in N$, provided that $E \in \Sigma$ and $\|\mu\|_p(E) < \delta$. Therefore (f_n) is uniformly absolutely continuous with respect to μ.

Definition 2.1-20. Let (f_n) be a sequence of μ-integrable functions. Then (f_n) is said to converge to a μ-integrable function f in L_1 if for every continuous semi-norm P the following is true:

$$\lim_n \int |f_n - f| dv(x^*\mu) = 0 \text{ uniformly in } x^* \in V^*,\ x^* \leq P.$$

Note that from Proposition 2.1-1 and 2.1-4 it follows that if f is μ-integrable and $\Phi(E) = \int_E f d\mu$, then $||\Phi||_p(E) < \infty$ for each E in Σ and each continuous semi-norm P.

Lemma 2.1-21. Let (f_n) be a sequence of μ-integrable functions. If (f_n) converges to a μ-integrable function f in L_1, then (f_n) is uniformly integrable.

Proof. Let P be an arbitrary but fixed continuous semi-norm. Since (f_n) converges to f in L_1,

$$\operatorname{limit}_n \int |f_n - f| dv(x^*\mu) = 0 \text{ uniformly in } x^* \in V^*,\ x^* \leq P.$$

Take $\varepsilon_o = 1$, then there is an integer n_1 such that

$$\int |f_n - f| dv(x^*\mu) < 1 \text{ for each } n \geq n_1,\ n \in N \text{ and } x^* \leq P.$$

Hence

$$\int |f_n| dv(x^*\mu) \leq \int |f_n - f| dv(x^*\mu) + \int |f| dv(x^*\mu)$$

$$\leq 1 + ||\Phi||_p(S) \text{ for each } n \in N,\ n \geq n_1, \text{ where}$$

$$||\Phi||_p(S) = \sup \{\int |f| dv(x^*\mu) : x^* \leq P\}.$$

Therefore

$$\sup \{\int |f_n| dv(x^*\mu) : x^* \leq P,\ n \in N\}$$

$$= \max \{1 + ||\Phi||_p(S),\ 1 + ||\Phi_1||_p(S),$$

$$1 + ||\Phi_2||_p(S), \ldots, 1 + ||\Phi_{n_1-1}||_p(S)\} < \infty,$$

where

$$||\Phi_k||_p(S) = \sup \{\int |f_k| dv(x^*\mu) : x^* \leq P\}$$

for each $k = 1, 2, \ldots, n_1 - 1$. This shows that (f_n) is L_1-bounded.

Let $\varepsilon > 0$ be given. Since (f_n) converges to f in L_1, we have,

$$\lim_n \int |f_n - f| dv(x^*\mu) = 0 \text{ uniformly in } x^* \le P.$$

There is an integer n_o such that

$\int |f_n - f| dv(x^*\mu) < \frac{\varepsilon}{2}$ for each $n \in N$, $n \ge n_o$ and each $x^* \le P$. Since f is μ-integrable, there is a number $\delta_o > 0$ such that $||\Phi||_p(E) < \frac{\varepsilon}{2}$ whenever $E \in \Sigma$ and $||\mu||_p(E) < \delta_o$.

Now,

$$\int_E |f_n| dv(x^*\mu) \le \int_E |f_n - f| dv(x^*\mu) + \int_E |f| dv(x^*\mu)$$

$$\le \int_E |f_n - f| dv(x^*\mu) + ||\Phi||_p(E) \text{ for each } x^* \le P,$$

therefore for every integer n, $n \ge n_o$ and every E in Σ with $||\mu||_p(E) < \delta_o$, we have

$$\int_E |f_n| dv(x^*\mu) < \varepsilon \text{ for each } x^* \le P.$$

Also because $f_1, \ldots, f_{n_o - 1}$ are μ-integrable, there exist numbers $\delta_1, \delta_2, \ldots, \delta_{n_o - 1}$ such that for each $m \in \{1, 2, \ldots, n_o - 1\}$, $||\Phi_m||_p(E) < \varepsilon$ whenever $E \in \Sigma$ and $||\mu||_p < \delta_m$. Choose $\delta = \min\{\delta_o, \delta_1, \ldots, \delta_{n_o - 1}\}$. Therefore we have $\int_E |f_n| dv(x^*\mu) < \varepsilon$ for each $x^* \le P$ and each $n \in N$, whenever $E \in \Sigma$ and $||\mu||_p(E) < \delta$. This means (f_n) is uniformly absolutely continuous with respect to μ. It follows from the preceding theorem that (f_n) is uniformly integrable.

Note that if (f_n) is an amart, then for each $n \in N$, f_n is μ-integrable. This fact follows from the definition of amart and Proposition 2.1-8.

Definition 2.1-22. Let (f_n) be a sequence of μ-measurable functions. Then we say (f_n) converges to a μ-measurable function f in μ-measure, or simply in measure, if for every $\varepsilon > 0$,

$$\lim_n \|\mu\|_P(\{s \in S: |f_n(s)-f(s)|>\varepsilon\}) = 0$$

for every continuous semi-norm P.

Definition 2.1-23. Let (f_n) be a sequence of μ-measurable functions. Then we say (f_n) converges to a μ-measurable function f μ-uniformly if for every $\varepsilon > 0$ and every continuous semi-norm P there exists a set A_ε in Σ such that $\|\mu\|_P(S-A_\varepsilon) < \varepsilon$ and (f_n) converges to f uniformly on A_ε.

The next theorem extends the Egoroff theorem to the case of a measure space (S, Σ, μ) where μ takes values in a locally convex topological vector space V.

Theorem 2.1-24. Let (f_n) be a sequence of μ-measurable functions from S to R. If (f_n) converges to a μ-measurable function f μ-a.e., then (f_n) converges μ-uniformly to f.

Proof. Let $\varepsilon > 0$ be given and let P be a continuous semi-norm. Suppose that E is the μ-null set such that $(f_n(s))$ converges to $f(s)$ for $s \notin E$. Let

$$E_{k,m} = \{s \in S: |f_r(s)-f(s)| < \frac{1}{m} \text{ for } r \geq k\},$$

then $E_{k+1,m} \supset E_{k,m}$. Since $(f_n(s))$ converges to $f(s)$ for $s \notin E$, we have $\bigcup_{k=1}^{\infty} E_{k,m} = S - E$ for all m. Therefore

$$E = \bigcap_{k=1}^{\infty} (S-E_{k,m}) = \operatorname*{limit}_{k} (S-E_{k,m}).$$

It follows from Theorem 2.1-2 that

$$||\mu||_p(E) = \operatorname*{limit}_{k} ||\mu||_p(S-E_{k,m}).$$

Since $||\mu||_p(E) = 0$, there is an integer k_m such that

$$||\mu||_p(S-E_{k_m,m}) < \frac{\varepsilon}{2^m} \text{ for each } m.$$

Let $A_\varepsilon = \bigcap_{m=1}^{\infty} E_{k_m,m}$, then $S - A_\varepsilon = \bigcup_{m=1}^{\infty} (S-E_{k_m,m})$.

It follows from Proposition 2.1-1 that

$$||\mu||_p(S-A_\varepsilon) \leq \sum_{m=1}^{\infty} ||\mu||_p(S-E_{k_m,m}) \leq \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} = \varepsilon$$

Given any $\delta > 0$, there is an integer m_o such that $\frac{1}{m_o} < \delta$. If $s \in A_\varepsilon$, then s lies in $E_{k_{m_o},m_o}$ also.

Therefore $|f_k(s)-f(s)| < \frac{1}{m_o} < \delta$ for each $k \geq k_{m_o}$.

Thus $(f_n(s))$ converges to $f(s)$ uniformly on A_ε. This proves that (f_n) converges to f μ-uniformly.

Lemma 2.1-25. Let (f_n) be a sequence of μ-measurable functions from S to R. If (f_n) converges to a μ-measurable function f μ-uniformly, then (f_n) converges to f in μ-measure.

Proof. Let $\varepsilon > 0$ be given and let P be a continuous semi-norm. Then there is an integer m such that $\varepsilon > \frac{1}{m}$. Since (f_n) converges to f μ-uniformly, given any number $\delta > 0$, there is a set A_δ in Σ such that $||\mu||_p(S-A_\delta) < \delta$

and (f_n) converges to f uniformly on A_δ. Consequently, there is an integer k_m such that $|f_k(s) - f(s)| < \frac{1}{m}$ for each $k \geq k_m$ and each $s \in A_\delta$. Since

$$\{s \varepsilon S : |f_k(s)-f(s)|>\varepsilon\} \subset \{s \varepsilon S: |f_k(s)-f(s)| \geq \frac{1}{m}\} =$$

$$S - \{s \varepsilon S: |f_k(s)-f(s)| < \frac{1}{m}\} \subset S-A_\delta \text{ for each } k \geq k_m.$$

Therefore $\|\mu\|_p\{s: |f_k(s)-f(s)| > \varepsilon\} \leq \|\mu\|_p(S-A_\delta) < \delta$

for each $k \geq k_m$.

Thus $\lim_k \|\mu\|_p\{s: |f_k(s)-f(s)| > \varepsilon\} = 0$.

This proves that (f_n) converges to f in μ-measure.

Theorem 2.1-26. Let (f_n) be a sequence of μ-measurable functions. If (f_n) converges μ-a.e. to a μ-measurable function f, then it converges to f in μ-measure.

Proof. The result obviously follows from Theorem 2.1-24 and Lemma 2.1-25.

Theorem 2.1-27. Let $(f_n, \Sigma_n, n \varepsilon N)$ be an amart and let V be a sequentially complete space. Then the following are equivalent:

(a) (f_n) converges to a μ-integrable function f in L_1;

(b) (f_n) is uniformly integrable.

Proof. Since $(f_n, \Sigma_n, n \varepsilon N)$ is an amart, we have that (f_n) is a sequence of μ-integrable functions. It follows from Lemma 2.1-21 that (a) implies (b). We need to show that (b) implies (a). Since (f_n) is uniformly integrable, by Theorem 2.1-19, we have that (f_n) is L_1-bounded. Also $(f_n, \Sigma_n, n \in N)$ is an amart. Therefore (f_n) converges to a μ-integrable function f μ-almost everywhere from Theorem 2.1-13. Therefore (f_n) converges to f in μ-measure. Thus for any $\varepsilon > 0$,

$$\underset{n}{\text{limit}} \, \|\mu\|_p(\{s\varepsilon S: |f_n(s)-f(s)| > \frac{\varepsilon}{3\|\mu\|_p(S)}\}) = 0$$

for each continuous semi-norm P.

(f_n) is uniformly integrable; hence (f_n) is uniformly absolutely continuous with respect to μ; that is,

$\underset{\|\mu\|_p(E)\to 0}{\text{limit}} \|\Phi_n\|_p(E) = 0$ uniformly in $n \in N$, where $\Phi_n(E) = \int_E f_n d\mu$ for each $n \in N$ and $E \in \Sigma$. Therefore there is a positive number η_1 such that for each $n \in N$, $\|\Phi_n\|_p(E) < \frac{\varepsilon}{3}$ whenever $E \in \Sigma$ and $\|\mu\|_p(E) < \eta_1$. Since f is μ-integrable, we have $\underset{\|\mu\|_p(E)\to 0}{\text{limit}} \|\Phi\|_p(E) = 0$ where $\Phi(E) = \int_E f d\mu$ for each $E \in \Sigma$. Therefore there is a positive number η_2 such that $\|\Phi\|_p(E) < \frac{\varepsilon}{3}$ whenever $E \in \Sigma$ and $\|\mu\|_p(E) < \eta_2$. Define $\eta = \min\{\eta_1, \eta_2\}$. Then there is an integer η_o such that

$$\|\mu\|_p(\{s\varepsilon S: |f_n(s)-f(s)| > \frac{\varepsilon}{3\|\mu\|_p(S)}\}) < \eta$$

for each $n \in N$, $n \geq \eta_o$. Define

$$A_n = \{s\varepsilon S: |f_n(s)-f(s)| > \frac{\varepsilon}{3\|\mu\|_p(S)}\}.$$

Therefore

$$\int |f_n - f|\, dv(x^*\mu) = \int_{S-A_n} |f_n - f|\, dv(x^*\mu) + \int_{A_n} |f_n - f|\, dv(x^*\mu)$$

$$\leq \frac{\varepsilon}{3\|\mu\|_p(S)} \cdot \|\mu\|_p(S-A_n) + \int_{A_n} |f_n|\, dv(x^*\mu) + \int_{A_n} |f|\, dv(x^*\mu)$$

$$\leq \frac{\varepsilon}{3} + \|\Phi_n\|_p(A_n) + \|\Phi\|_p(A_n) < \varepsilon \text{ for each } n \in N,$$

$n \geq n_o$ and each $x^* \leq P$, $x^* \in V^*$.

This proves that $\lim\limits_{n\to\infty} \int |f_n - f|\, dv(x^*\mu) = 0$ uniformly in $x^* \leq P$, $x^* \in V^*$. Thus (f_n) converges to f in L_1.

2. Decomposition of an Amart

In [7, p. 209] Edgar and Sucheston showed that a real-valued amart (X_n) for (Σ_n) integrable with respect to a probability measure P can be expressed as $X_n = Y_n + Z_n$, where Y_n is a martingale and Z_n is an amart with $Z_n \longrightarrow 0$ in L_1. In this section we develop this property for the real-valued amart (f_n) for (Σ_n) integrable with respect to a V-valued measure μ, where V is a locally convex topological vector space, provided V has the Radon-Nikodým property and some additional properties.

Let (S,Σ) be a measure space $\mu: S \longrightarrow V$ be a countably additive set function. We need to use the next theorem in the proof of the decomposition theorem for an amart.

Theorem 2.2-1 [6, p. 165]. Let (S,Σ,λ) be a measure space on Σ. Let Σ_1 be a sub-σ-field of Σ, and μ_1 be the restriction of μ from Σ to Σ_1. Then

(a) convergence in μ_1-measure implies convergence in μ-measure;

(b) a μ_1-null function is a μ-null function;

(c) a μ_1-null set is a μ-null set;

(d) a μ_1-measurable function is μ-measurable;

(e) a μ_1-integrable function f is μ-integrable and

$$\int_E f \, d\mu_1 = \int_E f \, d\mu \quad \text{for each } E \in \Sigma_1.$$

Since a function is μ-integrable if and only if it is $v(\mu)$-integrable, the result (e) of Theorem 2.2-1 still holds if λ is a real-valued measure.

Definition 2.2-2. A sequence of functions (f_n), $f_n\colon S \to R$ for each $n \in N$, adapted to an increasing sequence of sub-σ-fields (Σ_n) of Σ is defined to be a martingale if the following hold:

(1) (f_n) is a sequence of μ-integrable functions.

(2) $\operatorname{Sup} \{\int |f_n| dv(x^*\mu) : x^* \le P\} < \infty$ for each $n \in N$ and for every continuous semi-norm P.

(3) Given any $n \in N$ and any $E \in \Sigma_n$,

$$\int_E f_n d\mu = \int_E f_m d\mu \quad \text{holds for each } m \in N,\ m \ge n.$$

Definition 2.2-3. Let V be a locally convex topological vector space. Then V is said to have the Radon-Nikodým property if for every V-valued measure space (S, Σ, μ) and every V-valued measure ν such that

$\lim_{\|\mu\|_p(E)\to 0} \|\nu\|_p(E) = 0$ and $\|\nu\|_p(S) < \infty$ for every continuous semi-norm P, there is a μ-integrable function f from Σ to R such that $\nu(E) = \int_E f d\mu$ for every E in Σ. Note that throughout this section we will assume that V is sequentially complete and has the Radon-Nikodým property.

Let $(f_n, \Sigma_n, n \in N)$ be a real-valued amart in the measure space (S, Σ, μ), where μ is a V-valued measure. Since (f_n) is an amart, (f_n) is a sequence of μ-integrable functions. Define a function $\nu^n: \Sigma \longrightarrow V$ by $\nu^n(E) = \int_E f_n d\mu$ for each $E \in \Sigma$ and each $n \in N$ then $\lim_{\|\mu\|_p(E)\to 0} \|\nu^n\|_p(E) = 0$ and $\|\nu^n\|_p(S) < \infty$ from Theorems 2.1-1 and 2.1-4. Define a function $\nu_m^k: \Sigma_m \longrightarrow V$ by

$\nu_m^k(E) = \int_E f_k d\mu$ for $k, m \in N$ and each $E \in \Sigma_m$, then ν_m^k is is a countably additive set function on Σ_m. Since

$\lim_{\|\mu\|_p(E)\to 0} \|\nu^k\|_p(E) = 0$ for each continuous semi-norm P,

we have $\lim_{\|\mu\|_p(E)\to 0} \|\nu_m^k\|_p(E) = 0$. Now

$$v(x^*\nu_m^k, S) = \sup_{E_i \varepsilon \Sigma_m} \sum_{i=1}^{n} |x^*\nu_m^k(E_i)| = \sup_{E_i \varepsilon \Sigma_m} \sum_{i=1}^{n} \left|\int_{E_i} f_k dx^*\mu\right|$$

$$\leq \sup_{F_j \varepsilon \Sigma} \sum_{j=1}^{t} \left|\int_{F_j} f_k dx^*\mu\right| = \sup_{F_j \varepsilon \Sigma} \sum_{j=1}^{t} |x^*\nu^k(F_j)|$$

$$= v(x^*\nu^k, S), \text{ hence } \|\nu_m^k\|_p(S) \leq \|\nu^k\|_p(S) < \infty.$$

Since V has the Radon-Mikodým property, there is a μ/Σ_m-integrable, Σ_m-measurable function g_m^k such that

$$\nu_m^k(E) = \int_E g_m^k d\mu \quad \text{for each } E \in \Sigma_m.$$

But $\int_E f_k d\mu = \nu_m^k(E)$ for each $E \in \Sigma_m$, therefore

$$\int_E g_m^k d\mu = \int_E f_k d\mu \text{ for each } E \in \Sigma_m.$$

It follows that for $m, k \in N$ we have

$$\int_E g_m^k d\mu = \int_E g_n^k d\mu, \quad n \in N,\ n \geq m,\ E \in \Sigma_m.$$

Lemma 2.2-4. Assume that V is sequentially complete and has the Radon-Nikodým property. Let $(f_n, \Sigma_n, n \in N)$ be an L_1-bounded amart and $g_m^k = E(f_k/\Sigma_m)$ be given as before, then $(g_m^k, \Sigma_m^k \equiv \Sigma_m, k \in N)$ is an L_1-bounded amart and it converges to a μ/Σ_m-integrable function g_m μ/Σ_m-a.e.

Proof. Since $(f_n, \Sigma_n, n \in N)$ is an amart it follows that $\underset{\tau \in T}{\text{limit}} \int f_\tau d\mu$ exists in the sense that there is an element v_o in V such that

$$\underset{\tau \in T}{\text{limit}} \sup \{|x^*(\int f_\tau d\mu - v_o)| : x^* \leq P\} = 0$$

for every continuous semi-norm P. Let T_m be the collection of all bounded stopping times for $(\Sigma_m^k, k \in N)$ for each m, that is, $(\tau = t) \in \Sigma_m^t$ for each $t \in N$. If τ is a bounded stopping time in T_m with $\tau \geq m$, then $(\tau = t) \in \Sigma_m^t \subset \Sigma_t$ for each $t \geq m$; therefore τ is also a bounded stopping time in T. Let the integer r be an upper bound for τ, then

$$\int g_m^\tau d\mu = \sum_{t=m}^{r} \int_{(\tau=t)} g_m^t d\mu = \sum_{t=m}^{r} \int_{(\tau=t)} f_t d\mu = \int f_\tau d\mu.$$

Therefore,

$$\lim_{\tau \varepsilon T_m} \sup \{|x^*(\int g_m^\tau d\mu - v_o)| : x^* \leq P\}$$

$$= \lim_{\tau \varepsilon T} \sup \{|x^*(\int f_\tau d\mu - v_o)| : x^* \leq P\} = 0$$

for every continuous semi-norm P. Also

$$\sup \{\int |g_m^k| dv(x^*\mu) : x^* \leq P\} = \|v_m^k\|_P(S)$$

$\leq \|v^k\|_P(S) < \infty$ is true, therefore

$(g_m^k, \Sigma_m^k \equiv \Sigma_m, k \varepsilon N)$ is an amart. Since $(f_n, \Sigma_n, n \varepsilon N)$ is L_1-bounded,

$$\sup \{\int |g_m^n| dv(x^*\mu) : x^* \leq P, n \varepsilon N\}$$

$$= \sup \{\int |f_n| dv(x^*\mu) : x^* \leq P, n \varepsilon N\} < \infty.$$

Consequently, $(g_m^k, \Sigma_m^k \equiv \Sigma_m, k \varepsilon N)$ is an L_1-bounded amart. Therefore $(g_m^k, k \varepsilon N)$ converges to a μ/Σ_m-integrable, Σ_m-measurable function g_m μ/Σ_m-a.e. This proves the lemma.

Now let g_m be the μ/Σ_m-integrable, Σ_m-measurable function given in the preceding lemma. Then g_m is x^*/μ-integrable from Theorem 2.2-1. Since $(g_m^k, k \varepsilon N)$ converges to g_m μ/Σ_m-a.e. it follows, that

$$\|\mu/\Sigma_m\|_P(\{s \varepsilon S : g_m^k(s) \nrightarrow g_m(s)\}) = 0$$

for every continuous semi-norm P. Define $A = \{s \varepsilon S : g_m^k(s) \nrightarrow g_m(s)\}$ then $v(x^*\mu/\Sigma_m, A) = 0$ for each $x^* \varepsilon V^*$ and $A \varepsilon \Sigma_m$; hence $v(x^*\mu, A) = v(x^*\mu/\Sigma_m, A)$ and $\|\mu\|_P(A) = 0$. This proves that $(g_m^k, k \varepsilon N)$ converges to g_m μ-a.e.

Theorem 2.2-5. Let $(f_n, \Sigma_n, n \in N)$ be an L_1-bounded amart and g_m be a μ/Σ_m-integrable function such that $(E(f_k/\Sigma_m) : k \in N)$ converges μ-a.e. to g_m. If for each $m \in N$, g_m is μ-integrable and if there is a μ-integrable, Σ_1-measurable function g such that $|E(f_k/\Sigma_m)| \leq g$ for $m, k \in N$ the the following are true:

(1) $(g_m, \Sigma_m, m \in N)$ is a martingale.

(2) $(f_n, \Sigma_n, n \in N)$ can be uniquely expressed as the sum of a martingale and an amart.

The uniqueness is in the sense that if $f_n = g_n + h_n = g'_n + h'_n$ where $(g_n), (g'_n)$ are martingales and $(h_n), (h'_n)$ are amarts, then $g_n = g'_n$ μ-a.e. and $h_n = h'_n$ μ-a.e. for each integer n.

Proof. (1) Since g_m is μ-integrable it follows that

$$\sup \{\int |g_m| dv(x^*\mu) : x^* \leq P\} < \infty$$

for every continuous semi-norm P. Let n be an arbitrary integer and $E \in \Sigma_n$. Then $E \in \Sigma_m$ for each $m \geq n$, and

$$\int_E g_m d\mu = \lim_k \int_E E(f_k/\Sigma_m) d\mu =$$

$$\lim_k \int_E f_k d\mu = \lim_k \int_E E(f_k/\Sigma_n) d\mu = \int_E g_n d\mu$$

from Theorem 2.1-4. This shows that $(g_m, \Sigma_m, m \in N)$ is a martingale.

(2) Let $h_n = f_n - g_n$ for each $n \in N$. Now

$$\int |h_n| dv(x^*\mu) = \int |f_n - g_n| dv(x^*\mu)$$

$$\leq \int (|f_n| + |g_n|)\, dv(x^*\mu)$$

$$= \int |f_n| dv(x^*\mu) + \int |g_n| dv(x^*\mu).$$

Therefore,

$$\sup\{\int|h_n|dv(x^*\mu)\ :\ x^*\le P\}$$
$$\le \sup\{\int|f_n|dv(x^*\mu)\ :\ x^*\le P\}$$
$$+\ \sup\{\int|g_n|dv(x^*\mu)\ :\ x^*\le P\}<\infty.$$

Let τ be a bounded stopping time in T, and τ is bounded by an integer r, then

$$\int g_\tau d\mu = \sum_{t=1}^{r}\int_{(\tau=t)} g_t d\mu = \sum_{t=1}^{r}\int_{(\tau=t)} g_r d\mu$$
$$= \int g_r d\mu = \int g_1 d\mu.$$

Since $\lim_{\tau\varepsilon T}\int f_\tau d\mu$ exists, there is an element v of V such that

$$\lim_{\tau\varepsilon T}\sup\{|x^*(\int f\ d\mu-v)|\ :\ x^*\le P\}=0$$

for each continuous semi-norm P. Let $v_o = v - \int g_1 d\mu$ then $v_o \varepsilon V$. Therefore for each $x^*\ \varepsilon\ V^*$, $x^*\le P$ we have

$$x^*(\int h_\tau d\mu-v_o) = x^*(\int(f_\tau-g_\tau)d\mu-v_o)$$
$$= \int(f_\tau-g_\tau)dx^* - x^*(v_o) = \int f_\tau dx^*\mu - \int g_\tau dx^*\mu - x^*(v_o)$$
$$= x^*(\int f_\tau d\mu) - x^*(\int g_\tau d\mu) - x^*(v_o) = x^*(\int f_\tau d\mu)$$
$$- x^*(\int g_1 d\mu) - x^*(v_o) = x^*(\int f_\tau d\mu - \int g_1 d\mu - v_o)$$
$$= x^*(\int f_\tau d\mu - v).$$

Consequently,

$$\lim_{\tau\varepsilon T}\sup\{|x^*(\int h_\tau d\mu-v_o)|\ :\ x^*\le P\}$$
$$= \lim_{\tau\varepsilon T}\sup\{|x^*(\int f_\tau d\mu-v)|\ :\ x^*\le P\}=0.$$

Therefore, $(h_n, \Sigma_n, n \in N)$ is an amart. Assume that $f_n = h'_n + g'_n$ where $(h'_n, \Sigma_n, n \in N)$ is a martingale and $(g'_n, \Sigma_n, n \in N)$ is an amart. Let n be an arbitrary integer and $E \in \Sigma_n$, and x^* be an arbitrary element in V^*, then

$$\int_E g_m d\, x^*\mu = \int_E g_n d\, x^*\mu$$

for each $m \geq n$, $m \in N$, and

$$\operatorname*{limit}_m \int_E f_m d\, x^*\mu = \operatorname*{limit}_m \int_E E(f_m/\Sigma_n) dx^*\mu$$

$$= \int_E \operatorname*{limit}_m E(f_m/\Sigma_n) dx^*\mu = \int_E g_n\, dx^*\mu$$

from the Lebesque Dominated Convergence Theorem. Consequently,

$$\operatorname*{limit}_m \int_E h_m d\, x^*\mu = \operatorname*{limit}_m \int_E f_m d\, x^*\mu - \operatorname*{limit}_m \int_E g_m d\, x^*\mu$$

$$= \int_E g_n d\, x^*\mu - \int_E g_n d\, x^*\mu = 0$$

for each $E \in \Sigma_n$ and for each $n \in N$ and each $x^* \in V^*$. Similarly,

$$\operatorname*{limit}_m \int_E h'_m d\, x^*\mu = 0 \text{ for each } x^* \in V^* \text{ and } E \in \Sigma_n,\ n \in N.$$

Since $h_m + g_m = h'_m + g'_m$ for each $m \in N$,

$$\operatorname*{limit}_m \int_E h_m d\, x^*\mu + \operatorname*{limit}_m \int_E g_m dx^*,$$

$$= \operatorname*{limit}_m \int_E h'_m d\, x^*\mu + \operatorname*{limit}_m \int_E g'_m d\, x^*\mu \text{ for } E \in \Sigma_n,\ n \in N.$$

Therefore

$$\operatorname*{limit}_m \int_E g_m d\, x^*\mu = \operatorname*{limit}_m \int_E g'_m d\, x^*\mu$$

for $E \in \Sigma_n$, $n \in N$ and $x^* \in V^*$. But $\operatorname*{limit}_m \int_E g_m d\, x^*\mu = \int_E g_n d\, x^*\mu$ and $\operatorname*{limit}_m \int_E g'_m d\, x^*\mu = \int_E g'_n d\, x^*\mu$ for $E \in \Sigma_n$, $n \in N$ and each $x^* \in V^*$; therefore

$$\int_E g_n dx^*\mu = \int_E g'_n dx^*\mu$$

for each $E \varepsilon \Sigma_n$ and each $x^* \varepsilon V^*$. It follows that $g_n = g'_n$ $x^*\mu/\Sigma$-a.e. because g_n, g'_n are Σ_n-measurable. From Theorem 2.2-1 we have $g_n = g'_n$ $x^*\mu$-a.e. for every $x^* \varepsilon V^*$. Consequently, $g_n = g'_n$ μ-a.e. for each $n \varepsilon N$. This implies that $h_n = h'_n$ μ-a.e. for each $n \varepsilon N$. This completes the proof of the theorem.

3. Net Asymptotic Martingales

Definition 2.3-1. Let Λ be a directed set and $(f_\alpha, \alpha \varepsilon \Lambda)$ be a net of μ-measurable functions from S to R adapted to an increasing net $(\Sigma_\alpha, \alpha \varepsilon \Lambda)$ of sub-σ-fields of Σ. Then $(f_\alpha, \Sigma_\alpha, \alpha \varepsilon \Lambda)$ is called a net asymptotic martingale, or simply a net amart, if

$$\sup \{\int |f_\alpha| dv(x^*\mu) : x^* \leq P\} < \infty$$

holds for every $\alpha \varepsilon \Lambda$ and every continuous semi-norm P, and $\underset{\tau \varepsilon T}{\text{limit}} \int f_\tau d\mu$ exists in V, where T is the collection of all finite stopping times $\tau : S \rightarrow \Lambda$, which take only finitely many values in Λ and $(\tau = \alpha) \varepsilon \Sigma_\alpha$ for all $\alpha \varepsilon \Lambda$. The existence of this limit means that there is an element v in V such that for each $\varepsilon > 0$ and every continuous semi-norm P, there is a $\sigma \varepsilon T$ which depends on ε and P such that

$$\sup \{|x^*(\int f_\tau d\mu - v)| : x^* \leq P\} < \varepsilon$$

for each $\tau \varepsilon T$, $\tau \geq \sigma$.

Definition 2.3-2. Let $(f_\alpha, \Sigma_\alpha, \alpha \in \Lambda)$ be a net amart. Then $(f_\alpha, \Sigma_\alpha, \alpha \in \Lambda)$ is said to be uniformly convergent if for any $\varepsilon > 0$ there is an element $\sigma \in T$, which depends on ε only, such that for each $\tau \geq \sigma$, $\tau \in T$,

$$\sup \{|x^*(\int f_\tau d\mu - v)|: x^* \leq P\} < \varepsilon$$

for every continuous semi-norm P.

Lemma 2.3-3. Let Λ be a directed set and $(f_\alpha, \alpha \in \Lambda)$ be a net of functions defined from S to R. If $(f_\alpha, \alpha \in \Lambda)$ is Cauchy in μ-measure, that is,

$$\underset{\alpha,\alpha' \in \Lambda}{\text{limit}} \ (f_\alpha - f_{\alpha'}) = 0$$

in μ-measure, then there is a subsequence $(f_{\alpha_i}, i \in N)$ of $(f_\alpha, \alpha \in \Lambda)$, and a function f, such that $(f_{\alpha_i}, i \in N)$ converges to f μ-uniformly.

Proof. Let P be an arbitrary continuous semi-norm on V. Since $\underset{\alpha,\alpha' \in \Lambda}{\text{limit}} \ (f_\alpha - f_{\alpha'}) = 0$ in μ-measure, there is an element α_1 in Λ such that

$$\|\mu\|_p(s: |f_\alpha(s) - f_{\alpha'}(s)| > \tfrac{1}{2}) < \tfrac{1}{2}$$

for α, α' in Λ and $\alpha, \alpha' \geq \alpha_1$. Choose α_2 in Λ such that $\alpha_2 \geq \alpha_1$ and

$$\|\mu\|_p(s_\alpha: |f_\alpha(s) - f_{\alpha'}(s)| > \frac{1}{2^2}) < \frac{1}{2^2}$$

for α, α' in Λ and $\alpha, \alpha' \geq \alpha_2$. Therefore,

$$\|\mu\|_p(s: |f_{\alpha_2}(s) - f_{\alpha_1}(s)| < \tfrac{1}{2}) < \tfrac{1}{2} .$$

Continuing this process, we obtain a sequence $(\alpha_i, i \varepsilon N)$ such that

$$||\mu||_p(s: |f_\alpha(s)-f_{\alpha'}(s)| > \frac{1}{2^k}) < \frac{1}{2^k}$$

for α, α' in Λ; $\alpha, \alpha' \geq \alpha_k$, and

$$||\mu||_p(s: |f_{\alpha_{k+1}}(s)-f_{\alpha_k}(s)| < \frac{1}{2^k}) < \frac{1}{2^k}$$

for each $k \varepsilon N$. Define

$$E_i = (s: |f_{\alpha_i}(s)-f_{\alpha_{i+1}}(s)| > \frac{1}{2^i})$$

then $||\mu||_p(E_i) < \frac{1}{2^i}$ for each $i \varepsilon N$. Define

$$F_k = \bigcup_{i=k}^{\infty} E_i \text{ for each } k \varepsilon N, \text{ then } ||\mu||_p(F_k) \leq \frac{1}{2^{k-1}}.$$

We claim that $(f_{\alpha_i}, i \varepsilon N)$ is uniformly Cauchy on $S - F_k$. Let $\varepsilon > 0$ be given. Then there is an integer m such that $\varepsilon > \frac{1}{2^{m-1}}$ and $m > k$. Thus

$$|f_{\alpha_i}(s)-f_{\alpha_j}(s)| \leq \sum_{n=m}^{\infty} |f_{\alpha_n}(s)-f_{\alpha_{n+1}}(s)| < \frac{1}{2^{m-1}} < \varepsilon$$

for each $s \varepsilon S - F_k$ and $i,j \geq m$. This proves that (f_{α_i}) is uniformly Cauchy on $S - F_k$. Define $F = \bigcap_{k=1}^{\infty} F_k$. Then $F = \lim_k F_k$ because $(F_k, k \varepsilon N)$ is a decreasing sequence. Therefore, $||\mu||_p(F) = \lim_k ||\mu||_p(F_k) = 0$. Assume that $s \not\varepsilon F$, then $s \not\varepsilon F_k$ for some integer k. Therefore, $(f_{\alpha_i}(s), i \varepsilon N)$ is a Cauchy sequence in R. And there is a real-value a_s such that $(f_{\alpha_i}(s), i \varepsilon N)$ converges to a_s.

We define a function f, $f : S \longrightarrow R$, such that

$$f(s) = \begin{cases} 0 & \text{if } s \in F \\ \operatorname{limit}_i f_{\alpha_i}(s) & \text{if } s \in S-F. \end{cases}$$

Consequently, $(f_{\alpha_i}(s), i \in N)$ is a Cauchy sequence which converges to f uniformly on each of the sets $S - F_k$. This proves that $(f_{\alpha_i}, i \in N)$ converges to f μ-uniformly.

Lemma 2.3-4. Let $(f_\alpha, \alpha \in \Lambda)$ be a net of μ-measurable functions. If $(f_\alpha, \alpha \in \Lambda)$ is Cauchy in μ-measure, then there is a subsequence $(f_{\alpha_i}, i \in N)$ of $(f_\alpha, \alpha \in \Lambda)$, and a μ-measurable function f, such that $(f_{\alpha_i}, i \in N)$ converges to f μ-uniformly.

Proof. From the result of the Lemma 2.3-3 we have a subsequence $(f_{\alpha_i}, i \in N)$ of $(f_\alpha, \alpha \in \Lambda)$, and a function f, such that $(f_{\alpha_i}, i \in N)$ converges to f μ-uniformly. We claim that f is μ-measurable. Since (f_{α_i}) is a sequence of μ-measurable functions for each i in N, there is a sequence of μ-simple functions $(g_i^m, m \in N)$ such that $(g_i^m, m \in N)$ converges to f_{α_i} μ-a.e. Hence $(g_i^m, m \in N)$ converges to f_{α_i} μ-uniformly from the Theorem 2.1-24. Let P be a continuous semi-norm. Then there is a set A_k in Σ and an integer $m_k \in N$ such that

$$\|\mu\|_p(A_k) < \frac{1}{2^k} \quad \text{and} \quad |g_k^{m_k}(s) - f_{\alpha_k}(s)| < \frac{1}{2^k}$$

for each $s \in S-A_k$.

Define $h_k = g_k^{m_k}$ for each k, then $(h_k, k \in N)$ is a sequence of μ-simple functions. Define

$$B_m = \bigcup_{k=m}^{\infty} A_k, \quad B = \bigcap_{m=1}^{\infty} B_m, \text{ then } \|\mu\|_P(B) = 0.$$

Since $(f_{\alpha_i}, i \in N)$ converges to f μ-uniformly, (f_{α_i}) converges to f μ-a.e. Therefore, there is a set A such that $\|\mu\|_P(A) = 0$, and (f_{α_i}) converges to f on S-A. Thus $\|\mu\|_P(A \cup B) = 0$. We claim that $(h_k, k \in N)$ converges to f μ-a.e. Assume that $s \notin A \cup B$, then $s \notin A$ and $s \notin B$. Therefore $s \notin B_m$ for some $m \in N$ and hence $s \notin A_k$ for each $k \geq m$. Thus

$$|h_k(s)-f_{\alpha_k}(s)| < \frac{1}{2^k}$$

for each $k \geq m$. Since $s \notin A$, $(f_{\alpha_i}(s))$ converges to f(s). Let $\varepsilon > 0$ be given; then there exists an element α_t such that $t \geq m$, $t \in N$, $\frac{1}{2^t} < \frac{\varepsilon}{2}$ and

$$|f_{\alpha_i}(s)-f(s)| < \frac{\varepsilon}{2}$$

for each $i \geq t$, $i \in N$. Therefore, for each $i \geq t$ we have

$$|h_i(s)-f(s)| \leq |h_i(s)-f_{\alpha_i}(s)| + |f_{\alpha_i}(s)-f(s)| < \frac{1}{2^i} + \frac{\varepsilon}{2} < \varepsilon.$$

This shows that $(h_k, k \in N)$ converges μ-a.e. to f and f is a μ-measurable function.

Definition 2.3-5. Let $(f_\alpha, \alpha \in \Lambda)$ be a net of μ-measurable functions. Then $(f_\alpha, \alpha \in \Lambda)$ is said to converge to a μ-measurable function f in μ-measure if for each $\varepsilon > 0$,

$$\operatorname*{limit}_{\alpha\in\Lambda} \|\mu\|_P(\{s : s\in S, |f_\alpha(s)-f(s)| > \varepsilon\}) = 0,$$

for every continuous semi-norm P.

Lemma 2.3-6. Let $(f_\alpha, \alpha \in \Lambda)$ be a net of μ-measurable functions defined on S to R. Then $(f_\alpha, \alpha \in \Lambda)$ is Cauchy in μ-measure if and only if $(f_\alpha, \alpha \in \Lambda)$ converges to a μ-measurable function f in μ-measure.

Proof. It is obvious that the "if" part is true. Now we want to show that the "only if" part is true also.

Since $(f_\alpha, \alpha \in \Lambda)$ is Cauchy in μ-measure, there exists a subsequence $(f_{\alpha_i}, i \in N)$ of $(f_\alpha, \alpha \in \Lambda)$, and a μ-measurable function f such that $(f_{\alpha_i}, i \in N)$ converges to f in μ-measure. Let P be a continuous semi-norm on V and let $\varepsilon > 0$ be given. There is a number t such that $\frac{1}{2^t} < \varepsilon$,

$$||\mu||_p(s: |f_\alpha(s) - f_{\alpha_t}(s)| > \frac{1}{2^{t+1}}) < \frac{1}{2^{t+1}},$$

and

$$||\mu||_p(s: |f_\alpha(s) - f(s)| > \frac{1}{2^{t+1}}) < \frac{1}{2^{t+1}}$$

for each $\alpha \geq \alpha_t$. Therefore,

$$||\mu||_p(s: |f_\alpha(s) - f(s)| > \varepsilon) \leq ||\mu||_p(s: |f_\alpha(s) - f(s)| > \frac{1}{2^t})$$

$$\leq ||\mu||_p(s: |f_\alpha(s) - f_{\alpha_t}(s)| > \frac{1}{2^{t+1}})$$

$$+ ||\mu||_p(s: |f_{\alpha_t}(s) - f(s)| > \frac{1}{2^{t+1}})$$

$$< \frac{1}{2^t} < \varepsilon, \text{ for each } \alpha \in \Lambda, \alpha \geq \alpha_t.$$

This proves that $(f_\alpha, \alpha \in \Lambda)$ converges to f in μ-measure.

The proof of the next convergence theorem is modeled after a similar theorem in Edgar and Sucheston [7, p. 206] for the real-valued case.

Theorem 2.3-7. Let $(f_\alpha, \Sigma_\alpha, \alpha \in \Lambda)$ be a net amart. If $(f_\alpha, \Sigma_\alpha, \alpha \in \Lambda)$ is uniformly convergent and

$$\sup \{\int |f_\alpha| dv(x^*\mu) : x^* \leq P, \alpha \in \Lambda\} < \infty$$

for every continuous semi-norm P, then $(f_\alpha, \Sigma_\alpha, \alpha \in \Lambda)$ converges to a μ-measurable function f in μ-measure if V is sequentially complete.

Proof. From Lemma 2.3-5, we need only to show that $(f_\alpha, \Sigma_\alpha, \alpha \in \Lambda)$ is Cauchy in μ-measure. Assume that $(f_\alpha, \Sigma_\alpha, \alpha \in \Lambda)$ is not Cauchy in μ-measure. Then there is some $\varepsilon_o > 0$ and some continuous semi-norm P_o and some $\delta_o > 0$ such that for each $\alpha \in \Lambda$ there is an element $\alpha' \in \Lambda$ such that $\alpha' \geq \alpha$ and

$$||\mu||_{P_o}(\{s: |f_\alpha(s) - f_{\alpha'}(s)| > \varepsilon_o\}) \geq \delta_o.$$

Choose α_1 arbitrarily in Λ and let α_2 be an element in Λ such that $\alpha_2 \geq \alpha_1$ and

$$||\mu||_{P_o}(\{s: |f_{\alpha_2}(s) - f_{\alpha_1}(s)| > \varepsilon_o\}) \geq \delta_o.$$

Because $\lim_{\tau \in T} \int f_\tau d\mu$ converges uniformly in the sense that there is an element v in V such that for each $\eta > 0$, there is a finite stopping time σ in T, such that for each $\tau \in T$, $\tau \geq \sigma$,

$$\sup \{|x^*(\int f_\tau d\mu - v)|: x^* \leq P\} < \eta$$

holds for every continuous semi-norm P. Therefore, there is some $\sigma_o \in T$ such that

$$\sup \{|x^*(\int f_\tau d\mu - v)|: x^* \leq P\} < \frac{1}{3}$$

holds for every continuous semi-norm P and each $\tau \geq \sigma_o$, $\tau \in T$. Choose $\alpha_3 \in \Lambda$ such that α_3 is an upper bound of σ_o and $\alpha_3 \geq \alpha_2$. Continuing this process, we obtain a sequence $(\alpha_n, n \in N)$ such that if n is even, α_n is the one satisfying $\alpha_n \geq \alpha_{n-1}$ and

$$||\mu||_{P_o}(\{s: |f_{\alpha_n}(s) - f_{\alpha_{n-1}}(s)| > \varepsilon_o\}) \geq \delta_o;$$

if n is odd, then α_n is the one satisfying $\alpha_n \geq \alpha_{n-1}$ and

$$\sup \{|x^*(\int f_\tau d\mu - v)|: x^* \leq P\} < \frac{1}{n}$$

for every continuous semi-norm P and each $\tau \geq \alpha_n$, $\tau \in T$. It is obvious that $(f_{\alpha_n}, n \in N)$ is a sequence of μ-measurable functions adapted to $(\Sigma_{\alpha_n}, n \in N)$. We assert that $(f_{\alpha_n}, \Sigma_{\alpha_n}, n \in N)$ is an L_1-bounded amart. Since

$$\sup \{\int |f_\alpha| dv(x^*\mu): x^* \leq P, \alpha \in \Lambda\} < \infty$$

for every continuous semi-norm P, then

$$\sup \{\int |f_{\alpha_n}| dv(x^*\mu): x^* \leq P, n \in N\} < \infty$$

for every continuous semi-norm P.

Let $\varepsilon > 0$ be given. Then there is some $\alpha_{n_o} \in \Lambda$ such that n_o is even, $\varepsilon > \frac{1}{n_o}$ and for each $\tau \in T$, $\tau \geq \alpha_{n_o}$

$$\sup \{ | x^*(\int f_\tau d\mu - v) | : x^* \le P \} < \frac{1}{n_o} < \varepsilon$$

for every continuous semi-norm P. Therefore $(f_{\alpha_n}, \Sigma_{\alpha_n}, n \in N)$ is an L_1-bounded amart. Therefore $(f_{\alpha_n}, \Sigma_{\alpha_n}, n \in N)$ converges to a μ-integrable function μ-a.e. and hence $(f_{\alpha_n}, \Sigma_{\alpha_n}, n \in N)$ converges in μ-measure. But for every $n \in N$ we can always find some integer m, $m \ge n$ such that

$$\| \mu \|_{p_o}(\{s: |f_{\alpha_m}(s) - f_{\alpha_{m+1}}(s)| > \varepsilon_o) \ge \delta_o;$$

this implies that $(f_{\alpha_n}, \Sigma_{\alpha_n}, n \in N)$ is not Cauchy in μ-measure. Therefore, $(f_\alpha, \Sigma_\alpha, \alpha \in \Lambda)$ is Cauchy in μ-measure and hence $(f_\alpha, \Sigma_\alpha, \alpha \in \Lambda)$ converges to a μ-measurable function f in μ-measure from Lemma 2.3-5.

CHAPTER III

VECTOR-VALUED ASYMPTOTIC MARTINGALES INTEGABLE WITH RESPECT TO A VECTOR MEASURE

In [2] Bartle developed a theory of integration in which both the function to be integrated and the measure take values in a relatively general vector space. He considered there to be a continuous bilinear "multiplication" defined on the product of the vector spaces in which the function and the measure take their values, the product lying in a (possibly different) vector space.

In this chapter we shall first investigate a theory of martingales which take values in a Banach space, or B-space, and are integrable with respect to a measure whose values are in a (possibly different) Banach space. The integrals take values in a (possibly different) Banach space. Next we extend these properties to the amart case. The integrability here is in the sense of the integral defined in Bartle [2].

1. Basic Concepts and Notations

Let X and Y be real or complex normed vector spaces. Assume that there is a bilinear mapping, which is denoted by juxtaposition, defined on X × Y with values in a Banach space Z, satisfying $|xy|_Z \le K \, |x|_X \, |y|_Y$ for some fixed positive number K.

Let S be an abstract set and Σ be a field of subsets of S. Let μ be a finitely additive set function defined on Σ to Y. The semi-variation of μ is the extended non-negative function $\|\mu\|$ whose value on a set E in Σ, denoted by $\|E\|$ or $\|\mu\|(E)$, is defined to be

$$\|\mu\|(E) = \sup |\Sigma x_i \mu(E_i)|,$$

where the supremum is taken over all partitions of E into a finite number of disjoint subsets $(E_i) \subset \Sigma$ and all finite collections of elements $(x_i) \subset X$ with $|x_i| \leq 1$. The variation of μ is the extended non-negative function $|\mu|$, is defined by

$$|\mu| = \sup \Sigma|\mu(E_i)|$$

where the supremum is taken over all partitions of E into a finite number of disjoint sets $(E_i) \subset \Sigma$.

It is obvious that the semi-variation of μ is a monotone, subadditive function on Σ, and that the variation of μ is a monotone, additive function on Σ. Also, if E is in Σ, then

$$0 \leq \|\mu\|(E) \leq K \cdot |\mu|(E) \leq \infty.$$

We extend the definition of $\|\mu\|$ and $|\mu|$ to arbitrary subsets of S as follows: if $A \subset S$ then

$$\|A\| = \inf \{\|E\|: E \varepsilon \Sigma, A \subset E\}$$

and

$$|\mu|(A) = \inf \{|\mu|(E): E \varepsilon \Sigma, A \subset E\}.$$

It is evident that the extension of $\|\mu\|$ agrees with its former value on Σ and is a monotone, subadditive function on the collection of all subsets of S.

A subset E of S is a μ-null set if $\|E\| = 0$. A proposition holds μ-almost everywhere, abbreviated as μ-a.e., if it holds outside of a μ-null set. A μ-simple function is a function $f\colon S \to X$ which takes only finite many values $x_1,\ldots,x_n$, each non-zero value x_i being taken on a set E_i in Σ with $\|E_i\| < \infty$. Thus f can be represented as

$$f = \sum_{i=1}^{n} x_i I_{E_i}, \quad E_i \in \Sigma,$$

where I_{E_i} is the characteristic function of the set E_i.

We define the integral of a simple function

$$f = \sum_{i=1}^{n} x_i I_{E_i}, \quad E_i \in \Sigma \text{ over a set } E \text{ in } \Sigma \text{ by}$$

$$\int_E f d\mu = \sum_{i=1}^{n} x_i \mu(E \cap E_i).$$

It is easy to see that the integral of a simple function is independent of the representation given in its definition.

Lemma 3.1-1 [2, p. 340]. Let μ be a finitely aditive set function from Σ to Y.

(a) For each fixed E in Σ, the integral over E is a linear mapping defined on the linear space of simple functions on S to X, and has values in Z.

(b) For each fixed simple function, the integral is a finitely additive set function on Σ.

(c) If f is a simple function and $|f(s)| < M$ for all s in $E \varepsilon \Sigma$, then

$$\left|\int_T f d\mu\right| \leq M \, \|E\|.$$

A sequence (f_n) of functions on S to X is said to converge in μ-measure to a function f if

$$\|\{s \varepsilon S: |f_n(s)-f(s)| \geq \varepsilon\}\| \longrightarrow 0$$

as $n \longrightarrow \infty$ for each $\varepsilon > 0$.

A function $f: S \longrightarrow X$ is μ-measurable if it is the limit in measure of a sequence of simple functions. Clearly, the collection of all measurable functions on S to X is a linear space which is closed under the operation of convergence in measure of sequences. A sequence (f_n) of functions converges μ-almost uniformly to a function f on S if for every $\varepsilon > 0$ there is a subset A_ε of S such that $\|A_\varepsilon\| < \varepsilon$ and (f_n) converges to f uniformly on $S-A_\varepsilon$.

Lemma 3.1-2 [2, p. 340]. Let μ be a finitely additive set function on Σ to Y.

(a) μ-almost uniform convergence implies convergence in μ-measure to the same function.

(b) μ-almost uniform convergence implies μ-almost everywhere convergence to the same function.

We shall assume that μ is a finitely additive set function from S to Y from Definition 3.1-3 through Theorem 3.1-10.

Definition 3.1-3 [2, p. 341]. A function f, $f: S \rightarrow X$, is said to be μ-integrable over S if there is a sequence (f_n) of μ-simple functions on S to X such that the following conditions are true.

(1) The sequence (f_n) converges in μ-measure to f.

(2) The sequence (λ_n) of indefinite integrals

$$\lambda_n(E) \equiv \int_E f d\mu, \quad E \in \Sigma$$

has the property that given any $\varepsilon > 0$ there is a $\delta > 0$ such that if E is in Σ and $\|E\| < \delta$, then $|\lambda_n|(E) < \varepsilon$, $n = 1,2,\ldots$.

(3) The sequence (λ_n) has the property that given any $\varepsilon > 0$ there is a set E_ε in Σ with $\|E_\varepsilon\| < \infty$ and such that if F is in Σ and $F \subset S - E_\varepsilon$, then $|\lambda_n|(F) < \varepsilon$, $n = 1,2,\ldots$.

Condition (2) is frequently stated as "the sequence (λ_n) is uniformly absolutely continuous with respect to $\|\mu\|$," and (3) is stated as "(λ_n) is equicontinuous with respect to $\|\mu\|$."

Theorem 3.14 [2, p. 341]. If f is integrable over S in the sense of Definition 3.1-3, then for each E in Σ $\lim_{n\to\infty} \lambda_n(E)$ exists in the norm of Z. This limit is denoted by $\lambda(E)$ or by $\int_E f d\mu$ and is called the value of the indefinite integral λ at E, or the integral of f over the set E. In addition, the limit $\lambda(E) = \lim_n \lambda_n(E)$ exists in the norm of Z uniformly for E in Σ.

The integral is independent of the sequence of simple functions used to define it.

Theorem 3.1-5 [2 , p. 342]. (a) If E is in Σ then the set of functions integrable over E is a linear space and the integral over E is a linear mapping of this space into Z.

(b) If f is integrable over S, the integral of f is a finitely additive set function on the field Σ.

(c) If f is integrable over S, then

$$\lim_{\|E\|\to 0} \int_E f d\mu = 0.$$

(d) If f is integrable over S, then given any $\varepsilon > 0$ there is a set E_ε in Σ such that if $F \varepsilon \Sigma$ and $F \subset S-E$ then

$$\left|\int_F f d\mu\right| < \varepsilon.$$

Definition 3.1-6 [2 , p.342]. A function f, $f: S \longrightarrow X$, is μ-essentially bounded on a subset A of S if

$$\inf_N \sup_{s \varepsilon A-N} |f(s)| < \infty,$$

where the infimum is taken over all μ-null sets N. We write $\operatorname{ess\,sup}_{s \varepsilon A} |f(s)|$ for this number.

Theorem 3.1-7 [2 , p. 342]. If f is an essentially bounded measurable function then f is integrable over any set E in Σ with $\|E\| < \infty$ and

$$\left|\int_E f d\mu\right| \leq \{\operatorname{ess\,sup}_{s \varepsilon E} |f(s)|\} \|E\|.$$

Theorem 3.1-8. (VITALI CONVERGENCE THEOREM) [2, p. 343].

Let f be a function on S to X and let (f_n) be a sequence of integrable functions such that

(1) the sequence (f_n) converges in μ-measure to f;

(2) the sequence of indefinite integrals is uniformly absolutely continuous with respect to $\|\mu\|$;

(3) the indefinite integrals are equicontinuous with respect to $\|\mu\|$.

Then f is an integrable function and

$$\int_E f d\mu = \lim_n \int_E f_n d\mu, \quad E \in \Sigma.$$

Furthermore, the limit is uniform for E in Σ.

In the case when either X or Y is a scalar normed vector space, the integrability of a function f implies that of the function $|f(\cdot)|$; but this is not true here in general, as can be seen in the following example.

Example (a) [2 , p. 344]. Let X = Y = real Euclidean three-space, with the usual inner product, and let δ_1, δ_2 and δ_3 be the unit coordinate vectors. Let λ be Lebesgue measure on $S = [0,1]$ and let $\mu(E) = \lambda(E)\delta_1$, $g(s) \equiv s^{-1}\delta_2$. Then g is μ-integrable; but $|g(\cdot)|$ is not integrable.

Also the usual formulation of the Lebesgue Dominated Convergence Theorem is not true here in general, as can be seen in the following example.

Example (b) [2, p. 344]. Let S, X, Y, μ and g be as in Example (a). If $f_n(s) \equiv s^{-1+(1/n)} \delta_1$, then (f_n) is a sequence of integrable functions such that $|f_n(s)| \leq |g(s)|$ and (f_n) converges μ-a.e. and in μ-measure to $f_o(s) \equiv s^{-1}\delta_1$. But f_o is not integrable.

The converse of the Vitali Theorem fails in general. This can be shown in the following example.

Example (c) [2, p. 344]. Let S, X, Y, and μ be given as in Example (a). Let $h_n(s) \equiv 1\ \delta_2$ and $h_o(s) \equiv 1\ \delta_3$. Then each h_n is integrable and

$$\int_E h_o d\mu = \underset{n}{\text{limit}} \int_E h_n d\mu = 0$$

uniformly for $E \in \Sigma$. Conditions (2) and (3) of the Vitali Convergence Theoem are satisfied; but (h_n) does not converge to h_o at any point and not in measure.

Theorem 3.1-8 [2, p. 345] (DOMINATED CONVERGENCE THEOREM). Let (f_n) be a sequence of integrable functions on S to X which converges in measure to a function f. If there is an integer g such that for each E in Σ and $n \in N$, where N is the set of all natural numbers, we have

$$\left|\int_E f_n d\mu\right| \leq \left|\int_E g d\mu\right|,$$

then f is integrable on S and

$$\underset{n}{\text{limit}} \int_E f_n d\mu = \int_E f d\mu \text{ for each } E \in \Sigma.$$

Theorem 3.1-9 [2, p.345]. Let (f_n) be a sequence of μ-integrable functions from S to X such that (f_n) converges to a function f in measure. If $|f_n(s)| \leq M$ for almost all $s \in S$, then f is integrable over any set E in Σ with $\|E\| < \infty$ and

$$\underset{n}{\text{limit}} \int_E f_n d\mu = \int_E f d\mu.$$

Theorem 3.1-10 [2, p. 345]. Let (f_n) be a sequence of μ-integrable functions which converge almost uniformly to f. Then f is integrable over any set E in Σ with $\|E\| < \infty$ and

$$\underset{n}{\text{limit}} \int_E f_n d\mu = \int_E f d\mu.$$

From now on we shall assume that μ is a countably additive set function from σ-field Σ to Y. Under this assumption we can prove that if $f: S \longrightarrow X$ is μ-integrable, then the indefinite integral of f is a countably additive set function on Σ to Z.

Definition 3.1-11 [2, p. 346]. Let μ be a countably additive set function on a σ-field Σ to Y, then μ has the *-property (with respect to X) if there is a non-negative finite-valued countably additive measure ν on Σ such that $\nu(E) \longrightarrow 0$ if and only if $\|E\| \to 0$.

One can see that if μ has the *-property, then $\|S\| < \infty$ and $\|\mu\|$ is countably subadditive on subsets of S.

Theorem 3.1-12 [2, p. 346]. Assume that the countably additive set function μ has the *-property, then the following are true.

(a) If a sequence (f_n) of functions on S to X converges in μ-measure to a function f, then some subsequence converges μ-almost uniformly to f.

(b) If a sequence (f_n) converges μ-almost everywhere to f, then it converges μ-almost uniformly to f.

Theorem 3.1-13 [2, p. 347]. Assume that the countably additive set function μ has the *-property. Then a function f on S to X is integrable if and only if there is a sequence (f_n) of simple functions such that

(1) the sequence (f_n) converges to f almost everywhere;

(2) the sequence (λ_n) of indefinite integrals converges in the norm of Z for each E in Σ.

Theorem 3.1-14 [2, p. 347]. Assume that the countably additive set function μ has the *-property. Let (f_n) be a sequence of integrable functions such that

(1) the sequence (f_n) converges μ-a.e. to f;

(2) given $\varepsilon > 0$ there is a $\delta > 0$ such that is E in Σ and $|E|| < \delta$ then

$$\left|\int_E f_n d\mu\right| < \varepsilon \quad n \in N.$$

Then f is integrable on S and

$$\int_E f d\mu = \underset{n}{\text{limit}} \int_E f_n d\mu,$$

uniformly for $E \in \Sigma$.

2. Martingales

Throughout this section we shall assume that (S,Σ) is a measurable space; that is, S is a set and Σ is a σ-field of subsets of S; X, Y, Z are Banach spaces, $\mu: \Sigma \to Z$ is a countably additive set function, and $(\Sigma_n, n \in N)$ is a sequence of sub-σ-fields of Σ with $\Sigma = \sigma(\bigcup_{n=1}^{\infty} \Sigma_n)$.

Definition 3.2-1. Let (f_n) be a sequence of μ-measurable functions from S to X adapted to a sequence of sub-σ-fields $(\Sigma_n, n \in N)$. We say that $(f_n, \Sigma_n, n \in N)$ is a martingale if and only if the following are true.

(1) For each $n \in N$, f_n is a μ-integrable function.

(2) For every fixed n, $n \in N$,

$$\int_E f_n d\mu = \int_E f_m d\mu,$$

for each $E \in \Sigma_n$ and each $m \in N$, $m \geq n$.

A μ-measurable function τ, $\tau: S \to N$, is said to be a stopping time for (Σ_n) if $(\tau=n) \in \Sigma_n$ for each $n \in N$.

Definition 3.2-2. A class W of subsets of S is said to be a Π-system if it is closed under the formation of finite intersections:

(Π) $A, B \in W$ implies $A \cap B \in W$.

A class L is a λ-system if it contains S and is closed under the formation of proper differences and countable, increasing unions:

(λ_1) $S \in \mathcal{L}$;

(λ_2) $A, B \in \mathcal{L}$ and $A \subset B$ imply $B-A \in \mathcal{L}$;

(λ_3) $A_1, A_2, \ldots \in \mathcal{L}$ and $A_n \uparrow A$ imply $A \in \mathcal{L}$.

We remark that a class that is both a Π-system and a λ-system is a σ-field.

Theorem 3.2-3. (Dynkin's Π-λ Theorem) [3, p. 34]. If W is a Π-system and $\mathcal{L}$ is a λ-system, then $W \subset \mathcal{L}$ implies that $\sigma(W) \subset \mathcal{L}$.

Lemma 3.2-4. Let μ be a countably additive set function on Σ to Y and assume that there is a non-negative finite-valued countably additive measure ν on Σ such that $\|\mu\| << \nu$; that is, for each E in Σ $\nu(E) \rightarrow 0$ implies $\|\mu\|(E) \rightarrow 0$. Then each $B \in \sigma(\bigcup_{n=1}^{\infty} \Sigma_n) = \Sigma$ and for each $\varepsilon > 0$ there is a set $A_\varepsilon \in \bigcup_{n=1}^{\infty} \Sigma_n$ such that $\|\mu\|(A_\varepsilon \Delta B) < \varepsilon$.

Proof. Let $\mathcal{L}$ be the collection of all sets in Σ satisfying the following property:

> If E lies in $\mathcal{L}$, then for each $\varepsilon > 0$, there is a set E_ε in $\bigcup_{n=1}^{\infty} \Sigma_n$ such that $\nu(E \Delta E_\varepsilon) < \varepsilon$.

It is clear that $\bigcup_{n=1}^{\infty} \Sigma_n \subset \mathcal{L}$ and $\bigcup_{n=1}^{\infty} \Sigma_n$ is a Π-system. We claim that $\mathcal{L}$ is a λ-system.

(1) S is in $\mathcal{L}$ because S is in $\bigcup_{n=1}^{\infty} \Sigma_n$.

(2) Assume that A, B are in $\mathcal{L}$ and $A \subset B$. Let $\varepsilon > 0$ be given. There are A_ε, B_ε in $\bigcup_{n=1}^{\infty} \Sigma_n$ such that

$$\nu(B\Delta B_\varepsilon) < \frac{\varepsilon}{2} \text{ and } \nu(A\Delta A_\varepsilon) < \frac{\varepsilon}{2}.$$

Since $(B-A)\ \Delta\ (B_\varepsilon - A_\varepsilon) \subset (B\Delta B_\varepsilon) \cup (A\Delta A_\varepsilon)$,

$$\nu((B-A)\Delta(B_\varepsilon - A_\varepsilon)) \le \nu((B\Delta B_\varepsilon) \cup (A\Delta A_\varepsilon))$$

$$\le \nu(B\Delta B_\varepsilon) + \nu(A\Delta A_\varepsilon) < \varepsilon.$$

Also $B_\varepsilon - A_\varepsilon \in \bigcup_{n=1}^{\infty} \Sigma_n$ because $\bigcup_{n=1}^{\infty} \Sigma_n$ is a field.

Therefore B-A is in L.

(3) Assume that $A_1, A_2, \ldots,$ are in L and $A_n \uparrow A$. Let $B_n = A - A_n$, then $B_n \downarrow \Phi$. Since ν is a countably additive set function on Σ we have $\nu(B_n) \downarrow 0$.

Let $\varepsilon > 0$ be given. There is an integer n_o such that $\nu(B_{n_o}) < \frac{\varepsilon}{2}$. Since $A_{n_o} \in L$ there is a set

$$F_{n_o} \in \bigcup_{n=1}^{\infty} \Sigma_n \text{ such that } \nu(F_{n_o}\ \Delta\ A_{n_o}) < \frac{\varepsilon}{2}.$$

Now, $A\ \Delta\ F_{n_o} = (A - F_{n_o}) \cup (F_{n_o} - A) \subset (B_{n_o} \cup (A_{n_o} - F_{n_o}))$

$$\cup\ (F_{n_o} - A_{n_o}) = B_{n_o} \cup (A_{n_o}\ \Delta\ F_{n_o}).$$

Therefore, $\nu(A\ \Delta\ F_{n_o}) \le \nu(B_{n_o}) + \nu(A_{n_o}\ \Delta\ F_{n_o}) < \varepsilon$.

This implies $A \in L$. By Dynkin's Π-λ theorem we have $\sigma(\bigcup_{n=1}^{\infty} \Sigma_n) \subset L$. Therefore $L = \sigma(\bigcup_{n=1}^{\infty} \Sigma_n)$.

Since $||\mu|| \ll \nu$, for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\nu(E) < \delta$ implies $||\mu||(E) < \varepsilon$. Therefore for each B in $\sigma(\bigcup_{n=1}^{\infty} \Sigma_n)$ and for each $\varepsilon > 0$, there is a set A_ε in

$\bigcup_{n=1}^{\infty} \Sigma_n$ such that $\nu(B \Delta A_\varepsilon) < \delta$ and hence $\|B\Delta A_\varepsilon\| < \varepsilon$. This completes the proof of the lemma.

Theorem 3.2-5. Let μ be a countably additive set function on Σ to Y and ν be a non-negative finite-valued measure on Σ such that $\|\mu\| << \nu$. Assume that $(f_n, \Sigma_n, n \in N)$ is a martingale. If there is a μ-integrable function g such that

$$\left|\int_E f_n \, d\mu\right| \leq \left|\int_E g d\mu\right|,$$

for each E in Σ and each $n \in N$, then

$$\operatorname*{limit}_n \int_E f_n d\mu \text{ exists for each E in } \sigma(\bigcup_{n=1}^{\infty} \Sigma_n).$$

Proof. First assume that E is in $\bigcup_{n=1}^{\infty} \Sigma_n$; then E is in Σ_{n_o} for some integer n_o. From the definition of martingale we have

$$\int_E f_n \mu = \int_E f_{n_o} d\mu, \text{ for each } n \in N, n \geq n_o.$$

Therefore, $\operatorname*{limit}_n \int_E f_n d\mu = \int_E f_{n_o} d\mu$ for each $E \in \Sigma_{n_o}$.

Next we assume that $E \in \Sigma$ but $E \notin \bigcup_{n=1}^{\infty} \Sigma_n$. Let $\varepsilon > 0$ be given. Since g is μ-integrable, $\operatorname*{limit}_{\|B\| \to 0} \int_B g d\mu = 0$, there is a $\delta > 0$ such that $\|B\| < \delta$ implies $\left|\int_B g d\mu\right| < \frac{\varepsilon}{4}$. From lemma 3.2-4 there is a set A_ε in $\bigcup_{n=1}^{\infty} \Sigma_n$ such that $\|E\Delta A_\varepsilon\| < \delta$. A_ε is in Σ_{n_o} for some integer n_o; therefore

$$\int_{A_\varepsilon} f_n d\mu = \int_{A_\varepsilon} f_m d\mu \text{ for } n, m \in N, n, m \geq n_o.$$

Now let n, m be integers with $n, m \geq n_o$; then

$$\left|\int_E f_n d\mu - \int_E f_m d\mu\right| \leq \left|\int_E f_n d\mu - \int_{A_\varepsilon} f_n d\mu\right|$$

$$+ \left|\int_{A_\varepsilon} f_n d\mu - \int_E f_m d\mu\right| = \left|\int_E f_n d\mu - \int_{A_\varepsilon} f_n d\mu\right| + \left|\int_{A_\varepsilon} f_m d\mu - \int_E f_m d\mu\right| .$$

Also,

$$\left|\int_E f_n d\mu - \int_{A_\varepsilon} f_n d\mu\right| = \left|\int_{(E\cap A_\varepsilon)\cup(E-A_\varepsilon)} f_n d\mu - \int_{(E\cap A_\varepsilon)\cup(A_\varepsilon-E)} f_n d\mu\right|$$

$$= \left|\int_{E-A_\varepsilon} f_n d\mu - \int_{A_\varepsilon-E} f_n d\mu\right| \leq \left|\int_{E-A_\varepsilon} f_n d\mu\right| + \left|\int_{A_\varepsilon-E} f_n d\mu\right|$$

$$\leq \left|\int_{E-A_\varepsilon} g d\mu\right| + \left|\int_{A_\varepsilon-E} g d\mu\right| .$$

Since $||\mu||$ is a monotone function on Σ,

$$||\mu||(E-A_\varepsilon) \leq ||\mu||(E\Delta A) < \delta .$$

Therefore $\left|\int_{E-A_\varepsilon} g d\mu\right| < \frac{\varepsilon}{4}$.

This implies that

$$\left|\int_E f_n d\mu - \int_{A_\varepsilon} f_n d\mu\right| < \frac{\varepsilon}{2}.$$

Similarly, $\left|\int_E f_m d\mu - \int_{A_\varepsilon} f_m d\mu\right| < \frac{\varepsilon}{2}.$

Consequently, $\left|\int_E f_n d\mu - \int_E f_m d\mu\right| < \varepsilon$ for each $n, m \geq n_o$, $n, m \in N$.

This proves that $\{\int_E f_n d\mu,\ n \in N\}$ is a Cauchy sequence in Z. Therefore, $\lim_n \int_E f_n d\mu$ exists, because Z is a Banach space.

Lemma 3.2-6. Let T be the collection of all bounded stopping times for $(\Sigma_n, n \in N)$ and $(f_n, n \in N)$ be a sequence of μ-integrable functions adapted to a sequence $(\Sigma_n, n \in N)$. Then $(f_n, \Sigma_n, n \in N)$ is a martingale if and only if $\int f_\tau d\mu$ does not depend on τ, $\tau \in T$.

Proof. Assume that $(f_n, \Sigma_n, n \in N)$ is a martingale. Let τ_1, τ_2 be arbitrary stopping times in T,

$$\max_{s \in S} \tau_1(s) = m_1, \quad \max_{s \in S} \tau_2(s) = m_2 \text{ and } m_1 \geq m_2.$$

Then

$$\int f_{\tau_1} d\mu = \int_{\bigcup_{k=1}^{m_1} (\tau_1 = k)} f_{\tau_1} d\mu = \sum_{k=1}^{m_1} \int_{(\tau_1 = k)} f_k \, d\mu$$

$$= \sum_{k=1}^{m_1} \int_{(\tau_1 = k)} f_{m_1} d\mu = \int f_{m_1} d\mu.$$

$$\int f_{\tau_2} d\mu = \int_{\bigcup_{k=1}^{m_2} (\tau_2 = k)} f_{\tau_2} d\mu = \sum_{k=1}^{m_2} \int_{(\tau_2 = k)} f_k d\mu$$

$$= \sum_{k=1}^{m_2} \int_{(\tau_2 = k)} f_{m_1} d\mu = \int f_{m_1} d\mu.$$

Therefore

$$\int f_{\tau_1} d\mu = \int f_{\tau_2} d\mu \quad \text{for each } \tau_1, \tau_2 \in T.$$

This proves that $\int f_\tau \, d\mu$ does not depend on τ, $\tau \in T$. Conversely, assume that $\int f_\tau d\mu$ does not depend on $\tau \in T$. Let n be an arbitrary but fixed integer and B be a set in Σ_n. For each $m \in N$, $m \geq n$ define two functions τ_1, τ_2 on S to N as following:

$$\tau_1(s) = \begin{cases} n & \text{if } s \in B \\ m+1 & \text{if } s \in B^c, \end{cases}$$

$$\tau_2(s) = \begin{cases} m & \text{if } s \in B \\ m+1 & \text{if } s \in B^c. \end{cases}$$

Since $(\tau_1=n) = B \in \Sigma_n$ and $(\tau_1=m+1) = B^c \in \Sigma_n \subset \Sigma_{m+1}$, τ_1 is a bounded stopping time for $(\Sigma_n, n \in N)$.

Similarly τ_2 is a bounded stopping time for (Σ_n)

$$\int f_{\tau_1} d\mu = \int_B f_{\tau_1} d\mu + \int_{B^c} f_{\tau_1} d\mu = \int_B f_n d\mu + \int_{B^c} f_{m+1} d\mu$$

and $\displaystyle\int f_{\tau_2} d\mu = \int_B f_{\tau_2} d\mu + \int_{B^c} f_{\tau_2} d\mu = \int_B f_m d\mu + \int_{B^c} f_{m+1} d\mu.$

But $\displaystyle\int f_{\tau_1} d\mu = \int f_{\tau_2} d\mu$, hence $\displaystyle\int_B f_n d\mu = \int_B f_m d\mu.$

This proves that $(f_n, \Sigma_n, n \in N)$ is a martingale.

Lemma 3.2-7. Let $(f_n, \Sigma_n, n \in N)$ be a martingale and k be an arbitrary but fixed integer. Let A be a set in Σ_k, then $\displaystyle\int_A f_\tau d\mu = \int_A f_k d\mu$ for each $\tau \geq k$, $\tau \in T$.

Proof. Let τ be a bounded stopping time in T and $\max_{s \in S} \tau(s) = m$. Define a function $\tau_1 : S \to N$ by

$$\tau_1(s) = \begin{cases} \tau(s) & \text{if } s \in A \\ m+1 & \text{if } s \in A^c. \end{cases}$$

For each $n \in N$, $k \leq n \leq m$, $(\tau_1=n) = (\tau=n) \cap A \in \Sigma_n$ and $(\tau_1=m+1) = A^c \in \Sigma_k \subset \Sigma_{m+1}$; therefore τ_1 is a bounded stopping time for (Σ_n).

Define $\tau_2 : S \rightarrow N$ by

$$\tau_2(s) = \begin{cases} k & \text{for } s \in A \\ m+1 & \text{for } s \in A^c. \end{cases}$$

It is obvious that τ_2 is in T. Since

$$\int_A f_\tau d\mu - \int_A f_k d\mu = \int f_{\tau_1} d\mu - \int_{A^c} f_{m+1} d\mu$$

$$- \int f_{\tau_2} d\mu + \int_{A^c} f_{m+1} d\mu = \int f_{\tau_1} d\mu - \int f_{\tau_2} d\mu = 0.$$

Therefore,

$$\int_A f_\tau d\mu = \int_A f_k d\mu \text{ for each } \tau \geq k,\ \tau \in T.$$

Theorem 3.2-8. Let μ be a countably additive set function on Σ to Y and ν a non-negative finite-valued measure on Σ such that $||\mu|| << \nu$. If $(f_n, \Sigma_n, n \in N)$ is a martingale and there is a μ-integrable function g such that

$$|\int_B f_\tau d\mu| \leq |\int_B g d\mu|,$$

for each $\tau \in T$ and each $B \in \sigma(\bigcup_{n=1}^{\infty} \Sigma_n)$, then

$$\lim_{\tau \in T} \int_B f_\tau d\mu \text{ exists for each } B \in \sigma(\bigcup_{n=1}^{\infty} \Sigma_n).$$

Proof. First assume that B is a set in $\bigcup_{n=1}^{\infty} \Sigma_n$. Then B is in Σ_{n_o} for some $n_o \in N$. From Lemma 3.2-7 we have

$$\int_B f_\tau d\mu = \int_B f_{n_o} d\mu \quad \text{for each } \tau \in T,\ \tau \geq n_o.$$

Therefore

$$\lim_{\tau \in T} \int_B f_\tau d\mu = \int_B f_{n_o} d\mu .$$

Next assume that $B \in \Sigma$ and $B \notin \bigcup_{n=1}^{\infty} \Sigma_n$. By the same argument as in Theorem 3.2-5 and the fact that

$$\int_B f_\tau d\mu = \int_B f_k d\mu \text{ for each } B \in \Sigma_k,$$

for each $\tau \in T$, $\tau \geq k$, we can show that

$$\lim_{\tau \in T} \int_B f_\tau d\mu \text{ exists for each } B \notin \bigcup_{n=1}^{\infty} \Sigma_n .$$

Theorem 3.2-9. (Vitali-Hahn-Saks) [6, p. 158]. Let (S, Σ, λ) be a measure space, and (λ_n) a sequence of λ-continuous vector or scalar-valued additive set functions on Σ. If the limit, $\lim_n \lambda_n(E)$, exists for each E in Σ then

$$\lim_{v(\lambda,E)\to 0} \lambda_n(E) = 0, \text{ uniformly for } n = 1,2,\ldots .$$

In addition, if $v(\lambda,S) < \infty$, the function

$$F(E) = \lim_n \lambda_n(E) \text{ is countably additive on } \Sigma .$$

Theorem 3.2-10. Assume that μ is a countably additive set function on Σ to Y and that μ has the *-property. Let $(f_n, \Sigma_n, n \in N)$ be a martingale and g be a μ-integrable function such that

$$\left|\int_E f_n d\mu\right| \leq \left|\int_E g d\mu\right|,$$

for each $E \in \Sigma$ and each $n \in N$.

Let $\lambda_n(E) = \int_E f_n d\mu$ for each $E \in \Sigma$ and define $F(E) = \lim_n \lambda_n(E)$ (it is well defined from Theorem 3.2-5).

Then F is a countably additive set function on Σ to Z and $F << \|\mu\|$; that is, for $E \varepsilon \Sigma$ $\|E\| \rightarrow 0$ implies $F(E) \rightarrow 0$.

Proof. Since f_n is μ-integrable,

$$\lim_{\|E\| \to 0} \int_E f_n d\mu = 0; \text{ that is, } \lim_{\|E\| \to 0} \lambda_n(E) = 0.$$

By the hypothesis, μ has the *-property, and there is a non-negative, finite-valued set function ν on Σ such that $\|E\| \rightarrow 0$ if and only if $\nu(E) \rightarrow 0$. Therefore

$$\lim_{\nu(E) \to 0} \lambda_n(E) = 0 \quad \text{for each } E \varepsilon \Sigma.$$

Also

$$\lim_{n \to \infty} \lambda_n(E) = F(E) \text{ exists for each } E \varepsilon \Sigma.$$

From the Vitali-Hahn-Saks theorem we have that F is a countably additive set function on Σ to Z, and

$$\lim_{\nu(E) \to 0} \lambda_n(E) = 0 \text{ uniformly in } n \varepsilon N.$$

Let $\varepsilon > 0$ be given. There is a number $\delta > 0$ such that for each $E \varepsilon \Sigma$ with $\nu(E) < \delta$ implies

$$|\lambda_n(E)| < \frac{\varepsilon}{2}, \; n = 1,2,3,\ldots .$$

Let $E_o \varepsilon \Sigma$ and $\nu(E_o) < \delta$. Since $\lim_n \lambda_n(E_o) = F(E_o)$, there is an integer n_o such that

$$|\lambda_{n_o}(E_o) - F(E_o)| < \frac{\varepsilon}{2},$$

and hence $|F(E_o)| < |\lambda_{n_o}(E_o)| + \frac{\varepsilon}{2} < \varepsilon$. This proves that there is a number $\delta > 0$ such that for each E in Σ with $\nu(E) < \delta$ then $|F(E)| < \varepsilon$. Therefore $F << \nu$. But $\nu << \|\mu\|$ and hence $F << \|\mu\|$.

Corollary 3.2-11. Under the hypotheses of Theorem 3.2-10 and the additional hypotheses that $G(E) = \int_E g\,d\mu$ for each E in Σ and G has finite variation, F has finite variation.

Proof. From Theorem 3.2-10 we have that F is a countably additive set function on Σ to Z. Let E be in Σ,

$$v(F,E) = \sup \sum_{i=1}^{n} |F(E_i)|$$

where the supremum is taken over all partitions of E into a finite number of disjoint sets in Σ. Let $\{E_1,\ldots,E_n\}$ be an arbitrary but fixed finite partition of E; then

$$F(E_i) = \operatorname{limit}_n \int_{E_i} f_n\,d\mu .$$

There is an integer m_i such that

$$\left|\int_{E_i} f_{m_i}\,d\mu - F(E_i)\right| \le \frac{1}{n},$$

and hence

$$|F(E_i)| \le \left|\int_{E_i} f_{m_i}\,d\mu\right| + \frac{1}{n} \le \left|\int_{E_i} g\,d\mu\right| + \frac{1}{n},$$

for each $i = 1,2,\ldots,n$. Therefore,

$$\sum_{i=1}^{n} |F(E_i)| \le \sum_{i=1}^{n} \left|\int_{E_i} g\,d\mu\right| + 1 \le v(G,E) + 1.$$

This implies that $v(F,E) \le v(G,E) + 1 < \infty$. This proves that F has finite variation.

We shall extend the next two properties from the similar properties of the real-valued random variable case which were proved by Austin, Edgar and Tulcea in [1, p. 18].

Lemma 3.2-12. Let $(f_n, n \in N)$ be a sequence of X-valued, μ-measurable functions. Let $\Sigma_n = \sigma(f_1, f_2, \ldots, f_n)$ for each integer $n \geq 1$ and assume that

$$\Sigma = \sigma\left(\bigcup_{n=1}^{\infty} \Sigma_n\right).$$

Let h be an X-valued, μ-measurable function such that for each $s \in S$, $h(s)$ is a cluster value of the sequence $(f_n(s), n \in N)$.

If μ has the *-property, then given any $\varepsilon > 0$, $\delta > 0$, and integer $m \geq 1$, there is a bounded stopping time τ for $(\Sigma_n, n \in N)$ such that $\tau \geq m$ and

$$\|\mu\|(\{s: |f_{\tau(s)}(s) - h(s)| \geq \delta\}) \leq \varepsilon.$$

Proof. By the hypothesis h is a μ-measurable function, there is a sequence of μ-simple functions $(h_n, n \in N)$ such that $h_n \rightarrow h$ in μ-measure. Let $\varepsilon > 0$, $\delta > 0$, and integer $m \geq 1$ be given. There is a simple function h_{n_o} such that

$$\|\mu\| \ (s: |h_{n_o}(s) - h(s)| \geq \tfrac{\delta}{2}) < \tfrac{\varepsilon}{4}.$$

Define

$$E = \{s: |h_{n_o}(s) - h(s)| \geq \tfrac{\delta}{2}\}$$

and let
$$h_{n_o} = \sum_{k=1}^{m_o} x_k I_{E_k}$$

where $\bigcup_{k=1}^{m_o} E_k = E$, $E_k \in \Sigma$ and $x_k \in X$. From Lemma 3.2-4 we have that there are sets $A_1, A_2, \ldots, A_{m_o}$ in $\bigcup_{n=1}^{\infty} \Sigma_n$ such that $\|\mu\|(E_k \Delta A_k) < \frac{\varepsilon}{4m_o}$ and $A_k \in \Sigma_{n_k}$ for each $k = 1, 2, \ldots, m_o$.

Define $g_{n_o} = \sum_{k=1}^{m_o} x_k I_{A_k}$, $\bigcup_{k=1}^{m_o} (E_k \Delta A_k) = D$, and $M = \max \{m, n_1, n_2, \ldots, n_{m_o}\}$. Then g_{n_o} is Σ_M-measurable. We claim that

$$\{s: |g_{n_o}(s) - h(s)| \geq \tfrac{\delta}{2}\} \subset E \cup \left(\bigcup_{k=1}^{m_o} (A_k - E_k)\right).$$

Assume that $s_o \in \{s: |g_{n_o}(s) - h(s)| \geq \frac{\delta}{2}\}$ and $s_o \notin E$.

Now if s_o is in $E_k \cap A_k$ for some k, $1 \leq k \leq m_o$, then

$$h_{n_o}(s_o) = g_{n_o}(s_o) \text{ and hence } |g_{n_o}(s_o) - h(s_o)| < \tfrac{\delta}{2}.$$

This contradicts $s_o \in \{s: |g_{n_o}(s) - h(s)| \geq \frac{\delta}{2}\}$.

Therefore, $s_o \notin \bigcup_{k=1}^{m_o} (E_k \cap A_k)$.

But $\bigcup_{k=1}^{m_o} (E_k \cap A_k) \cup \left(\bigcup_{k=1}^{m_o} (A_k - E_k)\right) = S$;

then $s_o \in \bigcup_{k=1}^{m_o} (A_k - E_k)$.

This proves the assertion. Therefore,

$$\|\mu\|(\{s: |g_{n_o}(s) - h(s)| \geq \tfrac{\delta}{2}\})$$
$$\leq \|\mu\|(\{s: |h_{n_o}(s) - h(s)| \geq \tfrac{\delta}{2}\}$$
$$+ \sum_{k=1}^{m_o} \|\mu\|(A_k - E_k) \leq \frac{\varepsilon}{4} + m_o \cdot \frac{\varepsilon}{4m_o} = \frac{\varepsilon}{2}.$$

Since μ has the *-property, $\|S\| < \infty$. Therefore

$$\|\mu\|(\{s: |g_{n_o}(s) - h(s)| < \tfrac{\delta}{2}\}) > \|S\| - \tfrac{\varepsilon}{2}.$$

Since $h(s)$ is a cluster value of the sequence $(f_n(s), n \in N)$, we have

$$\{s: |h(s)-g_{n_o}(s)| < \tfrac{\delta}{2}\} \subset \{s: |f_n(s)-g_{n_o}(s)| < \tfrac{\delta}{2},$$

for some $n \geq M\} = \bigcup_{n=M}^{\infty} \{s: |f_n(s)-g_{n_o}(s)| < \frac{\delta}{2}\}$. Thus

$$\nu(\{s: |h(s)-g_{n_o}(s)| < \tfrac{\delta}{2}) \leq \nu(\bigcup_{n=M}^{\infty} \{s: |f_n(s)-g_{n_o}(s)| < \tfrac{\delta}{2}\}).$$

Define $B_k = \bigcup_{n=M}^{k} \{s: |f_n(s)-g_{n_o}(s)| < \frac{\delta}{2}\}$ for each $k \in N$, $k > M$, and $B = \bigcup_{n=M}^{\infty} \{s: |f_n(s)-g_{n_o}(s)| < \frac{\delta}{2}\}$. Then $\{B_k, k > M\}$ is an increasing sequence converging to B and hence

$$\nu(B-B_k) \to 0 \text{ as } k \to \infty,$$

and $||\mu||(B) \geq ||\mu||(\{s: |h(s)-g_{n_o}(s)| < \frac{\delta}{2}\}) > ||S|| - \frac{\varepsilon}{2}$.

Assume that $||\mu||(B_k) \leq ||S|| - \frac{\varepsilon}{2}$ for each $k \in N$, $k > M$.

Define $\alpha = \sup_{k>M} ||\mu||(B_k)$; then $\alpha < ||S|| - \frac{\varepsilon}{2}$.

Let $\varepsilon_o = ||S|| - \frac{\varepsilon}{2} - \alpha$; then $\varepsilon_o > 0$. Since μ has the *-property, there is a non-negative finite-valued measure ν on Σ such that $||\mu|| << \nu$ and hence there is a number $\delta > 0$ such that $\nu(G) < \delta$, $G \in \Sigma$ implies $||\mu||(G) < \varepsilon_o$. Since $\lim_k B-B_k = \phi$, there is an integer k_o such that $\nu(B-B_{k_o}) < \delta$. Therefore $||\mu||(B-B_{k_o}) < \varepsilon_o$,

$$||\mu||(B) < ||\mu||(B_{k_o}) + ||\mu||(B-B_{k_o}) < \alpha + \varepsilon_o = ||S|| - \tfrac{\varepsilon}{2}.$$

This contradicts $||\mu||(B) > ||S|| - \frac{\varepsilon}{2}$.

Therefore $\|\mu\|(B_{N'}) > \|S\| - \frac{\varepsilon}{2}$ for some $N' \in N$ and $N' > M$.

Define a function $\tau : S \longrightarrow N$ by

$$\tau(s) = \text{the first } n \text{ such that } M \leq n \leq N' \text{ and } |f_n(s) - g_{n_o}(s)| \leq \frac{\delta}{2}$$
$$= N' \text{ otherwise.}$$

Since g_{n_o} is Σ_M-measurable, if $n \in N$, $M < n < N'$, then

$(\tau = n) = \{s \in S: |f_k(s) - g_{n_o}(s)| > \frac{\delta}{2}$ for each integer k, $M \leq k < n\} \in \Sigma_{M-1} \subset \Sigma_n$. If $n = M$, then

$(\tau = M) = \{s: |f_M(s) - g_{n_o}(s)| \leq \frac{\delta}{2}\} \in \Sigma_M$. If $n = N'$, then

$(\tau = N') = \{s: |f_{N'}(s) - g_{n_o}(s)| \leq \frac{\delta}{2}\} \bigcup \{s: |f_n(s) - g_{n_o}(s)| > \frac{\delta}{2}$, for each n, $M \leq n \leq N'\} \in \Sigma_{N'}$. This implies that τ is a bounded stopping time for $(\Sigma_n, n \in N)$. Therefore,

$$\|\mu\|(\{s: |f_{\tau(s)}(s) - g_{n_o}(s)| \leq \frac{\delta}{2}\}) > \|S\| - \frac{\varepsilon}{2}$$

and hence $\|\mu\|(\{s: |f_{\tau(s)}(s) - g_{n_o}(s)| > \frac{\delta}{2}\}) < \frac{\varepsilon}{2}$.

Now $\{s: |f_{\tau(s)}(s) - h(s)| \geq \delta\} \subset \{s: |f_{\tau(s)}(s) - g_{n_o}(s)| > \frac{\delta}{2}\}$

$$\subset \{s: |g_{n_o}(s) - h(s)| \geq \frac{\delta}{2}\}.$$

Therefore

$$\|\mu\|(\{s: |f_{\tau(s)}(s) - h(s)| \geq \delta\})$$
$$\leq \|\mu\|(\{s: |f_{\tau(s)}(s) - g_{n_o}(s)| > \frac{\delta}{2}\})$$
$$+ \|\mu\|(\{s: |g_{n_o}(s) - h(s)| \geq \frac{\delta}{2}\}) < \varepsilon .$$

This completes the proof of the lemma.

Theorem 3.2-13. Assume that μ has the *-property. Let $(f_n, n \in N)$ be a sequence of X-valued, μ-measurable functions. Let $\Sigma_n = \sigma(f_1, f_2, \ldots, f_n)$ for each $n \in N$ and $\sigma(\bigcup_{n=1}^{\infty} \Sigma_n) = \Sigma$. If $h: S \to X$ is a μ-measurable function such that for each $s \in S$, $h(s)$ is a cluster value of the sequence $(f_n(s), n \in N)$, then the following are true.

(1) There is a strictly increasing sequence $(\tau_n, n \in N)$ of bounded stopping times for $(\Sigma_n, n \in N)$ such that $\lim_n f_{\tau_n}(s) = h(s)$, μ-a.e.

(2) If (f_n) is a sequence of μ-integrable functions, then h is μ-integrable and $\lim_n \int_E f_{\tau_n} d\mu = \int_E h d\mu$ for each $E \in \Sigma$.

Proof. (1) Let $\varepsilon > 0$, $\delta > 0$, $m \geq 1$ be given. From Lemma 3.2-12 there is a bounded stopping time τ for $(\Sigma_n, n \in N)$, $\tau \geq m$ such that

$$\|\mu\|(\{s: |f_{\tau(s)}(s) - h(s)| \geq \delta) \leq \varepsilon.$$

We obtain an increasing sequence of bounded stopping times $(\tau_n, n \in N)$ as follows:

When $n = 1$, we choose $\tau_1 \in T$ such that $\tau_1 \geq 1$ and $\|\mu\|(\{s: |f_{\tau_1(s)}(s) - h(s)| \geq 1) \leq 1$;

when $n = 2$, we choose $\tau_2 \in T$ such that $\tau_2 \geq \max\{\tau_1(s): s \in S\}$ and $\|\mu\|\{s: |f_{\tau_2(s)}(s) - h(s)| \geq \frac{1}{2}\} \leq \frac{1}{2}$;

by induction, we have that for every n,

$$\tau_n \geq \max \{\tau_{n-1}(s): s\varepsilon S\} \text{ and}$$

$$||\mu|| \{s: |f_{\tau_n(s)}(s)-h(s)| \geq \tfrac{1}{n}\} \leq \tfrac{1}{n}.$$

We claim that $(f_{\tau_n}, n \varepsilon N)$ converges to h in μ-measure.

Let $\varepsilon > 0$, $\delta > 0$ be given. There is an integer n_o such that $\delta > \frac{1}{n_o}$ and $\varepsilon > \frac{1}{n_o}$. If $n \varepsilon N$, $n \geq n_o$, then

$$||\mu|| (\{s: |f_{\tau_n}(s)-h(s)| \geq \delta\})$$

$$\leq ||\mu|| (\{s: |f_{\tau_n}(s)-h(s)| \geq \tfrac{1}{n}\}) \leq \tfrac{1}{n} < \varepsilon.$$

This proves the assertion. Since μ has the *-property from Theorem 3.1-12, there is a subsequence $(\tau_{k_n}, n \varepsilon N)$ such that $(f_{\tau_{k_n}}, n \varepsilon N)$ converges μ-almost uniformly to h, and therefore converges μ-almost everywhere to h.

(2) Since $(f_n, n \varepsilon N)$ is a sequence of μ-integrable functions, $(f_{\tau_{k_n}}, n \varepsilon N)$ is a sequence of μ-integrable functions which converge μ-almost uniformly to h. Since μ has the *-property, $||\mu||(S) < \infty$. Therefore, from Theorem 3.1-10, h is a μ-integrable function over every set E in Σ and

$$\lim_n \int_E f_{\tau_{n_k}} d\mu = \int_E Y d\mu.$$

In [1], Austin, Edgar and Tulcea used the same property of the preceding theorem to show that a sequence of real-valued amarts comverges almost everywhere to a function under some conditions. We cannot extend this convergence property here, because it is not true in general that if two μ-integrable functions f and g satisfy $\int_B fd\mu = \int_B gd\mu$ for each B in Σ then $f = g$ μ-a.e. This can be shown in the following.

Example. Let X = Y = real Euclidean three-space, with the usual inner product. Let δ_1, δ_2 and δ_3 be the unit coordiate vectors and λ be Lebesque measure $S = [0,1]$ and $\mu(E) = \lambda(E)\delta_1$. Let $A = [0,\frac{1}{2}]$, define two functions f, g from S to X by

$$f = I_{A^c}\,\delta_3 + I_A\,\delta_1$$

$$g = I_{A^c}\,\delta_2 + I_A\,\delta_1.$$

Then for every B in Σ we have

$$\int_B fd\mu = \lambda(A \cap B) = \int_B gd\mu$$

But for each $s\varepsilon A^c$, $f(s) \neq g(s)$; and $\|A^c\| = \frac{1}{2}$.

Theorem 3.2-14. Assume that μ has the *-property. Let $(f_n, \Sigma_n, n \varepsilon N)$ be a martingale, $\Sigma_n = \sigma(f_1,\ldots f_n)$ for every $n \varepsilon N$ and $\Sigma = \sigma(\bigcup_{n=1}^{\infty} \Sigma_n)$. Let $h: S \rightarrow X$ be a a μ-measurable function such that for each s in S, h(S) is a cluster value of the sequence $(f_n(s), n \varepsilon N)$. If there is a μ-integrable function g such that

$$\left|\int_E f_\tau d\mu\right| \leq \left|\int_E gd\mu\right|,$$

for every $\tau \varepsilon T$ and each $E \varepsilon \Sigma$ then

$$\lim_{\tau\varepsilon T} \int_E f_\tau d\mu = \int_E hd\mu \quad \text{for each E in } \Sigma.$$

Proof. From Theorem 3.2-8, we have that $\lim_{\tau \in T} \int_E f_\tau d\mu$ exists for each E in Σ. Also from Theorem 3.2-13, we have that there is an increasing sequence of stopping times $(\tau_n, n \in N)$ such that for each E in Σ

$$\lim_n \int_E f_{\tau_n} d\mu = \int_E h d\mu.$$

Therefore, $\lim_{\tau \in T} \int_E f_\tau d\mu = \int_E h d\mu, \quad E \in \Sigma.$

3. Asymptotic Martingales

In this section we shall continue to use the same notation used in Sections 1 and 2.

Definition 3.3-1. Let (f_n) be a sequence of μ-measurable functions from S to X adapted to a sequence of sub-σ-fields $(\Sigma_n, n \in N)$. We say that $(f_n, \Sigma_n, n \in N)$ is an asymptotic martingale, or amart, if and only if the following are true.

(1) For every n in N, f_n is a μ-integrable function.

(2) The limit,

$$\lim_{\tau \in T} \int f_\tau d \quad \text{exists in } Z,$$

where T is the collection of all bounded stopping times for $(\Sigma_n, n \in N)$.

From Lemma 3.2-6, we have that every martingale is an amart.

Now we shall investigate several properties for vector-valued amarts which are integrable with respect to vector-valued measure.

Proposition 3.3-2. Let $(f_n, \Sigma_n, n \in N)$ be an amart and $||S|| < \infty$. Then

$$\sup \{|\int f_\tau d\mu|: \tau \in T\} < \infty.$$

Proof. First we want to show that if f is a μ-integrable function from S to X then

$$\sup \{|\int_E fd\mu|: E \in \Sigma\} < \infty.$$

Assume that f is a μ-integrable function on S to X. From Theorem 3.1-4, there is a sequence of μ-simple functions $(g_n, n \in N)$ on S to X such that

$$\int_E fd\mu = \operatorname*{limit}_n \int_E g_n d\mu, \text{ uniformly for E in } \Sigma.$$

Therefore, there is an integer N_1 such that for every E in Σ,

$$|\int_E g_{N_1} d\mu - \int_E fd\mu| < 1.$$

Thus,

$$|\int_E fd\mu| < 1 + |\int_E g_{N_1} d\mu| < 1 + M \cdot ||S|| < \infty$$

where $\quad M = \max \{|g_{N_1}(s)|: s \in S\} < \infty$

from Theorem 3.1-7. This proves the assertion.

Now, (f_n) is an amart, and there is an integer n_o such that

$$|\int f_\tau d\mu - \int f_{n_o} d\mu| < 1, \ \tau \in T, \ \tau \geq n_o.$$

Therefore,

$$|\int f_\tau d\mu| < 1 + |\int f_{n_o} d\mu| \leq 1 + M_o$$

where $\quad M_o = \sup\{|\int_E f_{n_o} d\mu|: E \in \Sigma\} \cdot ||S||.$

If τ is a bounded stopping time with $\tau \leq N_2$, then

$$\left|\int f_\tau d\mu\right| = \left|\sum_{k=1}^{N2} \int_{(\tau=k)} f_k d\mu\right| \leq \sum_{k=1}^{N_2} \left|\int_{(\tau=k)} f_k d\mu\right|$$

$$\leq \sum_{k=1}^{N_2} (1 + M_k),$$

where $M_k = \sup \{|\int_E f_k d\mu| : E \varepsilon \Sigma\} \cdot \|S\|$.

Let $K = \max \{1+M_o, 1+M_1, 1+M_2, \ldots, 1+M_{N_2}\}$

then $\sup \{|\int f_\tau d\mu| : \tau \varepsilon T\} < K < \infty$.

Theorem 3.3-3. Assume that μ has the $*$-property. Let $(f_n, \Sigma_n, n \varepsilon N)$ be an amart and $\Sigma = \sigma(\bigcup_{n=1}^{\infty} \Sigma_n)$. If there is a μ-integrable function g such that

$$\left|\int_E f_\tau d\mu\right| \leq \left|\int_E g d\mu\right|, \; E \varepsilon \Sigma, \; \tau \varepsilon T,$$

then for every E in Σ $\lim_{\tau \varepsilon T} \int_E f_\tau d\mu$ exists.

Proof. Since g is μ-integrable, let $\varepsilon > 0$ be given; then there is a number $\delta > 0$ such that E in Σ and $\|E\| < \delta$ implies

$$\left|\int_E g d\mu\right| < \frac{\varepsilon}{6}.$$

Let $E \varepsilon \Sigma$ be given. Since μ has the $*$-property, there is a set A in some sub-σ-field Σ_{n_o} such that $\|\mu\|(A \Delta E) < \delta$ from Lemma 3.2-4. Since $(f_n, n \varepsilon N)$ is an amart, there is an integer n_1 such that

$$\left|\int f_{\tau_1} d\mu - \int f_{\sigma_1} d\mu\right| < \frac{\varepsilon}{3}, \; \tau_1, \sigma_1 \varepsilon T, \; \tau_1 , \sigma_1 \geq n_1.$$

Define $n_2 = \max(n_o, n_1)$. Let τ, σ be arbitrary bounded stopping times in T, $\tau, \sigma \geq n_2$. Define τ_1, σ_1 from S to N by

$$\begin{aligned} \tau_1(s) &= \tau(s) && \text{for } s \in A \\ &= n_2 && \text{for } s \notin A \end{aligned}$$

and

$$\begin{aligned} \sigma_1(s) &= \sigma(s) && \text{for } s \in A \\ &= n_2 && \text{for } s \notin A. \end{aligned}$$

Then $(\tau_1 = n_2) = ((\tau = n_2) \cap A) \cup A^c \in \Sigma_{n_2}$,

and $(\tau_1 = m) = (\tau = m) \cap A \in \Sigma_m$, $m \in N$, $m \geq n_2$; because A is in Σ_{n_2}. This implies that τ_1 is a bounded stopping time. Similarly, σ_1 is also a bounded stopping time. But

$$\begin{aligned} \left|\int_E f_\tau d\mu - \int_E f_\sigma d\mu\right| &\leq \left|\int_E f_\tau d\mu - \int_A f_\tau d\mu\right| \\ &+ \left|\int_A f_\tau d\mu - \int_A f_\sigma d\mu\right| + \left|\int_A f_\sigma d\mu - \int_E f_\sigma d\mu\right| \\ &\leq \left|\int_{E-A} f_\tau d\mu - \int_{A-E} f_\tau d\mu\right| + \left|\int f_{\tau_1} d\mu - \int f_{\sigma_1} d\mu\right| \\ &+ \left|\int_{E-A} f_\sigma d\mu - \int_{A-E} f_\sigma d\mu\right| \\ &\leq 2\left|\int_{E-A} g d\mu\right| + 2\left|\int_{A-E} g d\mu\right| + \frac{\varepsilon}{3} \leq \varepsilon. \end{aligned}$$

This proves that for each E in Σ

$$\lim_{\tau \in T} \int_E f_\tau d\mu \quad \text{exists.}$$

Theorem 3.3-4. Assume that μ has the *-property. Let (f_n) be a sequence of μ-integrable functions adapted to an increasing sequence $(\Sigma_n, n \in N)$ of sub-σ-fields of Σ. If (f_n) converges μ-almost everywhere to a μ-measurable function f, then $(f_n, \Sigma_n, n \in N)$ is an amart.

Proof. Let $(\tau_n, n \in N)$ be an arbitrary increasing sequence of bounded stopping times for $(\Sigma_n, n \in N)$. Since $(f_n, n \in N)$ converges μ-almost everywhere to f, $(f_{\tau_n}, n \in N)$ converges μ-a.e. to f also. Since μ has the *-property, from Theorem 3.1-12, $(f_{\tau_n}, n \in N)$ converges μ-almost uniformly to f.

From 3.1-10, f is integrable over S and

$$\int f d\mu = \lim_n \int f_{\tau_n} d\mu,$$

because μ has the *-property, $\|S\| < \infty$. This implies that

$$\lim_{\tau \in T} \int f_\tau d\mu = \int f d\mu.$$

Therefore, $(f_n, \Sigma_n, n \in N)$ is an amart.

Theorem 3.3-5. (The optional sampling theorem for amarts.) Let $(f_n, \Sigma_n, n \in N)$ be an amart and $(\tau_k, k \in N)$ be a non-decreasing sequence of bounded stopping times for $(\Sigma_n, n \in N)$. Define $g_k = f_{\tau_k}$ and $J_k = \Sigma_{\tau_k} = \{A \in \Sigma :$

$$A \cap (\tau_k = n) \in \Sigma_n \text{ for all } n\}.$$

Assume that μ has the *-property. Then $(g_k, J_k, k \in N)$ is an amart.

Proof. We shall complete the proof in the following three steps.

Step 1. We claim that g_k is a J_k-measurable, μ-integrable function for each $k \in N$. Let k be an arbitrary integer in N, $m_o = \max \{\tau_k(s) : s \in S\}$ and

$$A_n = (\tau_k = n), \; n = 1,2,\ldots,m_o,$$

then $A_n \varepsilon \Sigma_n$, $n = 1,2,\ldots,m_o$. For each n, $n = 1,2,\ldots,m_o$, there is a sequence of μ-simple functions $(h_n^m, m \varepsilon N)$ such that $(h_n^m, m \varepsilon N)$ converges to f_n in μ-measure, and the indefinite integrals of $\{h_n^m, m \varepsilon N\}$ are uniformly absolutely continuous and equicontinuous with respect to $||\mu||$.

Let $\delta > 0$, $\varepsilon > 0$ be given. There is an integer M such that if $m \geq M$, then

$$||\mu||(\{s: |h_n^m(s) - f_n(s)| > \delta\}) < \frac{\varepsilon}{m_o}, \quad n = 1,2,\ldots,m_o.$$

Define a sequence of simple functions $(g_k^m, m \varepsilon N)$ on S to X as follows:

$$g_k^m(s) = h_n^m(s) \quad \text{if } s \varepsilon A_n, \ n = 1,2,\ldots,m_o.$$

Since
$$\begin{aligned} A \cap (\tau_k = t) &= A_n \quad \text{if } t = n \\ &= \phi \quad \text{if } t \neq n, \end{aligned}$$

$$A_n \varepsilon J_k \quad \text{for every } n = 1,2,\ldots,m_o.$$

Thus g_k^m is a J_k-measurable function. Also, for each m, $m \geq M$,

$$\begin{aligned} &||\mu||(\{s: |g_k^m(s) - g_k(s)| > \delta\}) \\ &= ||\mu||(\{s: |g_k^m(s) - f_{\tau_k}(s)| > \delta\}) \\ &= ||\mu||(\bigcup_{n=1}^{m}\{s: |h_n^m(s) - f_n(s)| > \delta\}) \\ &\leq \sum_{n=1}^{m_o} ||\mu||(\{s: |h_n^m(s) - f_n(s)| > \delta\}) < \varepsilon. \end{aligned}$$

Therefore, $(g_k^m, m \varepsilon N)$ converges to g_k in μ-measure.

Define $\lambda_m(E) = \int_E g_k^m d\mu$ for each $m \in N$ and $E \in \Sigma$. We claim that $(\lambda_m, m \in N)$ is uniformly absolutely continuous with respect to $\|\mu\|$ and $(\lambda_m, m \in N)$ is equicontinuous with respect to $\|\mu\|$.

Since the indefinite integrals of $(h_n^m, m \in N)$ are uniformly absolutely continuous with respect to $\|\mu\|$, given any $\varepsilon > 0$, there is a number $\delta > 0$ such that F in Σ and $\|\mu\|(F) < \delta$ implies

$$\left|\int_F h_n^m \, d\mu\right| < \frac{\varepsilon}{m_o}, \quad n = 1,2,\ldots,m_o, \quad m \in N.$$

Let E be a set in J_k with $\|E\| < \delta$, then

$$\|\mu\|(E \cap A_n) < \delta, \quad n = 1,2,\ldots,m_o.$$

Therefore,

$$\left|\int_E g_k^m d\mu\right| \le \sum_{n=1}^{m_o} \left|\int_{E \cap A_n} h_n^m d\mu\right| < \varepsilon, \; m \in N.$$

This implies that $(\lambda_m, m \in N)$ is uniformly absolutely continuous with respect to $\|\mu\|$.

Also the indefinite integrals of $\{h_n^m, m \in N\}$ are equicontinuous with respect to $\|\mu\|$. For any $\varepsilon > 0$, there are sets E_ε^n, $n = 1,2,\ldots,m_o$, in Σ such that $\|E_\varepsilon^n\| < \infty$, and if $F \subset S - E_\varepsilon^n$, $F \in \Sigma$, then

$$\left|\int_F h_n^m d\mu\right| < \frac{\varepsilon}{m_o}, \quad m \in N, \; n = 1,2,\ldots,m_o.$$

Define $E_\varepsilon = \bigcup_{n=1}^{m_o} E_\varepsilon^n$; then $\|E_\varepsilon\| < \infty$, and if F in J_k, $F \subset S - E_\varepsilon$, then $F \subset S - E_\varepsilon^n$, $F \in \Sigma$, for every $n = 1,2,\ldots,m_o$ and

$$|\int_F g_k^m d\mu| \leq \sum_{n=1}^{m_o} |\int_{F\cap A_n} h_n^m d\mu| \leq \varepsilon, \ m \in N.$$

This proves that $(\lambda_m, m \in N)$ is equicontinuous with respect to $||\mu||$. Therefore, g_k is a J_k-measurable, μ-integrable function on S.

Step 2. We claim that if σ is a stopping time for $(J_k, k \in N)$, then τ_σ is a stopping time for $(\Sigma_n, n \in N)$.

Let σ be a stopping time for $(J_k, k \in N)$, $(\sigma = k) \in J_k$. Therefore,

$$((\tau_k = n) \cap (\sigma = k) \in \Sigma_n, \ k, n \in N.$$

Thus $(\tau_\sigma=n) = \bigcup_{k=1}^{n_o} ((\tau_k=n) \cap (\sigma=k)) \in \Sigma_n, \ n \in N.$

This proves that τ_σ is a stopping time for $(\Sigma_n, n \in N)$.

Step 3. We claim that $(g_k, J_k, k \in N)$ is an amart. Let $\varepsilon > 0$ be given. There is some integer n_o such that

$$|\int f_\tau d\mu - \int f_{\tau'} d\mu| < \frac{\varepsilon}{2},$$

for any two-bounded stopping times τ, τ' for $(\Sigma_n, n\in N)$, $\tau, \tau' \geq n_o$. Let $\tau_\infty = \operatorname{limit}_k \tau_k$; then τ_∞ is a stopping time for $(\Sigma_n, n \in N)$ and for each $N_1 \in N$

$$f_{\tau_k \wedge N_1} \longrightarrow f_{\tau_\infty \wedge N_1} \quad \text{as } k \longrightarrow \infty.$$

Since μ has the *-property, from Theorem 3.3-4, $(f_{\tau_k \wedge N_1}, J_k, k \in N)$ is an amart.

We choose an integer K such that for every bounded stopping times σ, σ' for $(J_k, k \in N)$ with $\sigma, \sigma' \geq K$,

$$\left|\int f_{\tau_\sigma \wedge N_1} d\mu - \int f_{\tau_{\sigma'} \wedge N_1} d\mu\right| < \frac{\varepsilon}{2}.$$

Let σ, σ' be bounded stopping times for $(J_k,\ k \in N)$ and $\sigma, \sigma' \geq K$. Now

$$\int f_{\tau_\sigma \vee N_1} d\mu + \int f_{\tau_\sigma \wedge N_1} d\mu$$

$$= \left(\int_{(\tau_\sigma \geq N_1)} f_{\tau_\sigma} d\mu + \int_{(\tau_\sigma < N_1)} f_{N_1} d\mu\right) + \left(\int_{(\tau_\sigma \geq N_1)} f_{N_1} d\mu + \int_{(\tau_\sigma < N_1)} f_{\tau_\sigma} d\mu\right)$$

$$= \int f_{\tau_\sigma} d\mu + \int f_{N_1} d\mu.$$

Therefore,

$$\left|\int g_\sigma d\mu - \int g_{\sigma'} d\mu\right| = \left|\int f_{\tau_\sigma} d\mu - \int f_{\tau_{\sigma'}} d\mu\right|$$

$$= \left|\int f_{\tau_\sigma \vee N_1} d\mu + f_{\tau_\sigma \wedge N_1} d\mu - \int f_{\tau_{\sigma'} \vee N_1} d\mu - \int f_{\tau_{\sigma'} \wedge N_1} d\mu\right|$$

$$\leq \left|\int f_{\tau_\sigma \vee N_1} d\mu - \int f_{\tau_{\sigma'} \vee N_1} d\mu\right| + \left|\int f_{\tau_\sigma \wedge N_1} d\mu - \int f_{\tau_{\sigma'} \wedge N_1} d\mu\right|$$

$$< \varepsilon.$$

This proves that $(g_k,\ J_k,\ k \in N)$ is an amart.

Corollary 3.3-6. Assume that μ has the *-property. If $(f_n,\ \Sigma_n,\ n \in N)$ is an amart and σ is a stopping time for $(\Sigma_n,\ n \in N)$, then $(f_{n\wedge\sigma},\ J_n,\ n \in N)$ is an amart, where

$$J_n = \{A \in \Sigma:\ A \cap (n \wedge \sigma = k) \in \Sigma_k,\ k \in N\}.$$

Proof. Let $\tau_n = n \wedge \sigma$, $n \in N$; then $(\tau_n,\ n \in N)$ is a nondecreasing bounded stopping time for $(\Sigma_n,\ n \in N)$. From Theorem 3.3-5 $(f_{n\wedge\sigma},\ J_n,\ n \in N)$ is an amart.

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